Flexural-Torsional Buckling and Vibration Analysis of Composite Beams

E.J. Sapountzakis\(^1\) and G.C. Tsiatas\(^2\)

Abstract: In this paper the general flexural-torsional buckling and vibration problems of composite Euler-Bernoulli beams of arbitrarily shaped cross section are solved using a boundary element method. The general character of the proposed method is verified from the formulation of all basic equations with respect to an arbitrary coordinate system, which is not restricted to the principal one. The composite beam consists of materials in contact each of which can surround a finite number of inclusions. It is subjected to a compressive centrally applied load together with arbitrarily transverse and/or torsional distributed or concentrated loading, while its edges are restrained by the most general linear boundary conditions. The resulting problems are (i) the flexural-torsional buckling problem, which is described by three coupled ordinary differential equations and (ii) the flexural-torsional vibration problem, which is described by three coupled partial differential equations. Both problems are solved employing a boundary integral equation approach. Besides the effectiveness and accuracy of the developed method, a significant advantage is that the method can treat composite beams of both thin and thick walled cross sections taking into account the warping along the thickness of the walls. The proposed method overcomes the shortcoming of possible thin tube theory (TTT) solution, which its utilization has been proven to be prohibitive even in thin walled homogeneous sections. Example problems of composite beams are analysed, subjected to compressive or vibratory loading, to illustrate the method and demonstrate its efficiency and wherever possible its accuracy. Moreover, useful conclusions are drawn from the buckling and dynamic response of the beam.

Keyword: Flexural-torsional buckling; Flexural-torsional vibration; Composite beam; Boundary integral equation; Analog equation method; Free vibrations; Forced vibrations

1 Introduction

Composite beams have been increasingly used in recent years as structural members due to their high strength/stiffness properties for light weight materials. The design of such structures subjected to compressive or vibratory loading, presents a serious challenge and necessitates a reliable and accurate analysis. This becomes much more complicated in the case the beam’s cross section centroid does not coincide with its shear center (asymmetric beams), leading to the formulation of the flexural-torsional buckling or vibration problem of composite beams, respectively.


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pultruded fiber-reinforced plastic composite open channel or cantilever I-beams using the Rayleigh-Ritz method. Moreover, the flexural-torsional buckling and vibration problems have been studied by many researchers in laminated composite beams, such as Matsunaga (2001) employing Hamilton’s principle, Vinogradov and Derrick (2000) based on analytic solutions and Song and Waas (1997), Lellep and Sakkov (2006) in stepped rectangular beams of simple boundary conditions. However, the aforementioned formulations concerning composite beams of thin walled cross sections or laminated cross-sections are analyzing these beams with respect to cross section mid lines ignoring the warping along the thickness of the walls. Moreover, they do not satisfy the continuity conditions of transverse shear stress at layer interfaces and assume that the transverse shear stress along the thickness coordinate remains constant, leading to the fact that kinematic or static assumptions cannot be always valid (Karama, Afaq and Mistou, 2003; Reddy, 1989; Touratier, 1992). To the authors’ knowledge publications on the solution to the general flexural-torsional buckling or vibration problem of arbitrarily shaped composite cross sections do not exist.

In this investigation a boundary element method is developed for the general flexural-torsional buckling and vibration analysis of composite Euler-Bernoulli beams of arbitrarily shaped cross section. The composite beam consists of materials in contact, each of which can surround a finite number of inclusions, with modulus of elasticity $E_j$ and shear modulus $G_j$, occupying the regions $\Omega_j$ ($j = 1, 2, \ldots, K$) of the $y, z$ plane (Fig. 1). The materials of these regions are assumed homogeneous, isotropic and linearly elastic. Let also the boundaries of the nonintersecting regions $\Omega_j$ be denoted differential equations. Among many respective Boundary Element and Meshless Methods such as the Meshless Regularized Integral Equation Method (MRIEM) (Liu, 2007), the Meshless Local Petrov-Galerkin (MLPG) Method (Andreaus, Batra and Porfiri, 2005) and the Dual Boundary Element Method (Purbolaksono and Aliabadi, 2005) the authors applied the Analog Equation Method (AEM) (Katsikadelis, 2002a) for the solution of the aforementioned problems. According to this method, the three coupled fourth order differential equations are replaced by three uncoupled ones subjected to fictitious load distributions under the same boundary conditions. Besides the effectiveness and accuracy of the developed method, a significant advantage is that the method can treat composite beams of both thin and thick walled cross sections taking into account the warping along the thickness of the walls. The method overcomes the shortcoming of possible thin tube theory (TTT) solution, which its utilization has been proven to be prohibitive even in thin walled homogeneous sections (Sapountzakis and Tsiatas, 2007a). Moreover, the method permits the inclusion of the torsion-bending and flexural coupling stiffnesses which play an important role in the response of the beam and have to be taken always into account (Sapountzakis and Tsiatas, 2007a). Example problems of composite beams are analysed to illustrate the method and demonstrate its efficiency and wherever possible its accuracy. Moreover, useful conclusions are drawn from the buckling and dynamic response of the beam.

2 Statement of the problem

Let us consider an initially straight Euler-Bernoulli beam of length $l$ (Fig. 1), of constant arbitrary cross-section of area $A$. The cross section consists of materials in contact, each of which can surround a finite number of inclusions, with modulus of elasticity $E_j$ and shear modulus $G_j$, occupying the regions $\Omega_j$ ($j = 1, 2, \ldots, K$) of the $y, z$ plane (Fig. 1). The materials of these regions are assumed homogeneous, isotropic and linearly elastic. Let also the boundaries of the nonintersecting regions $\Omega_j$ be denoted
by \( \Gamma_j \) \((j = 1, 2, \ldots, K)\). These boundary curves are piecewise smooth, i.e. they may have a finite number of corners. In Fig. 1a \( CYZ \) and \( SYZ \) are coordinate systems (not necessarily principal) through the cross section’s centroid \( C \) and shear center \( S \), respectively. Moreover, \( y_C, z_C \) are the coordinates of the centroid \( C \) with respect to \( SYZ \) system of axes. The beam is subjected to a compressive centrally applied load together with arbitrarily transverse and/or torsional distributed or concentrated loading, while its edges are restrained by the most general linear boundary conditions. The two considered problems are:

(i) The flexural-torsional buckling problem, which is described by the following three coupled ordinary differential equations

\[
E_1 I_Z \frac{d^4 v}{dx^4} + E_1 I_{YZ} \frac{d^4 w}{dx^4} + P \left( \frac{d^2 v}{dx^2} - z_C \frac{d^2 \theta}{dx^2} \right) = p_Y \tag{1}
\]

\[
E_1 I_Y \frac{d^4 w}{dx^4} + E_1 I_{YZ} \frac{d^4 v}{dx^4} + P \left( \frac{d^2 w}{dx^2} + y_C \frac{d^2 \theta}{dx^2} \right) = p_Z \tag{2}
\]

\[
E_1 C_S \frac{d^4 \theta}{dx^4} - G_1 l \frac{d^2 \theta}{dx^2} + P \left( r_\omega \frac{d^2 \theta}{dx^2} - z_C \frac{d^2 v}{dx^2} + y_C \frac{d^2 w}{dx^2} \right) = m_x + p_Z y_C - p_Y z_C \tag{3}
\]
subjected to the following boundary conditions

\[ \alpha_1 v(x) + \alpha_2 R_Y(x) = \alpha_3 \] (4a)

\[ \alpha_1 \frac{dv(x)}{dx} + \alpha_2 M_Z(x) = \alpha_3 \] (4b)

\[ \beta_1 w(x) + \beta_2 R_Z(x) = \beta_3 \] (5a)

\[ \beta_1 \frac{dw(x)}{dx} + \beta_2 M_Y(x) = \beta_3 \] (5b)

\[ \gamma_1 \theta(x) + \gamma_2 M_t(x) = \gamma_3 \] (6a)

\[ \gamma_1 \frac{d\theta(x)}{dx} + \gamma_2 M_t(x) = \gamma_3 \] (6b)

at the beam ends \( x = 0, l \), where \( v = v(x), w = w(x) \) are the deflections of the shear center along \( y, z \) axes, respectively; \( \theta(x) \) is the angle of twist of the cross-section about the shear center \( S \); \( I_Y, I_Z, I_{YZ} \) are the moments and the product of inertia with respect to the centroid \( C \) given as

\[ I_Z = \sum_{j=1}^{K} \frac{E_j}{E_1} I_{Z_j}, \] (7a)

\[ I_Y = \sum_{j=1}^{K} \frac{E_j}{E_1} I_{Y_j}, \] (7b)

\[ I_{YZ} = \sum_{j=1}^{K} \frac{E_j}{E_1} I_{YZ_j}, \] (7c)

with \( I_{Y_j}, I_{Z_j}, I_{YZ_j} \) \( (j = 1, 2, \ldots, K) \) are the moments and the product of inertia of the materials, while, \( i_{\omega} \) is the polar radius of inertia with respect to the shear center \( S \) (Kollar, 2001a)

\[ i_{\omega}^2 = z_C^2 + y_C^2 + \frac{E_1 I_Z + E_1 I_Y}{\sum_{j=1}^{K} \frac{E_j}{E_1} A_j} \] (8)

Moreover,

\[ C_S = \sum_{j=1}^{K} \frac{E_j}{E_1} C_{S_j}, \] (9a)

\[ I_i = \sum_{j=1}^{K} \frac{G_j}{G_1} I_j \] (9b)

are the warping and torsion rigidities of the composite cross section, with \( C_{S_j} \) and \( I_{ij}, (j = 1, 2, \ldots, K) \), the corresponding constants of the \( \Omega_j \) region, respectively, (Sauer, 1980) which are established using a BEM procedure (Katsikadelis, 2002b).

In the boundary conditions (4), (5) \( R_Y, M_Y \) and \( R_Z, M_Z \) are the reactions and bending moments with respect to \( Y \) and \( Z \) axes, respectively, given as

\[ R_Y = -E_1 I_Y \frac{d^2 w(x)}{dx^2} - E_1 I_{YZ} \frac{d^3 v(x)}{dx^3} \] (10)

\[ M_Y = -E_1 I_Y \frac{d^2 w(x)}{dx^2} - E_1 I_{YZ} \frac{d^3 v(x)}{dx^3} \] (11)

\[ R_Z = -E_1 I_Z \frac{d^2 w(x)}{dx^2} + E_1 I_{YZ} \frac{d^3 w(x)}{dx^3} \] (12)

\[ M_Z = E_1 I_Z \frac{d^2 v(x)}{dx^2} + E_1 I_{YZ} \frac{d^3 w(x)}{dx^3} \] (13)

(ii) The flexural-torsional vibration problem, which is described by the following three coupled partial differential equations

\[ E_1 I_Z \frac{\partial^4 v}{\partial x^4} + E_1 I_{YZ} \frac{\partial^4 w}{\partial x^4} + \rho_1 A \left( \frac{\partial^2 v}{\partial t^2} - z_C \frac{\partial^2 \theta}{\partial t^2} \right) = p_y \] (14)

\[ E_1 I_Y \frac{\partial^4 w}{\partial x^4} + E_1 I_{YZ} \frac{\partial^4 v}{\partial x^4} + \rho_1 A \left( \frac{\partial^2 w}{\partial t^2} + y_C \frac{\partial^2 \theta}{\partial t^2} \right) = p_z \] (15)

\[ E_1 C_S \frac{\partial^4 \theta}{\partial x^4} - G_1 I_i \frac{\partial^2 \theta}{\partial x^2} + \rho_1 I_s \frac{\partial^2 \theta}{\partial t^2} \]

\[ + \rho_1 A \left( -z_C \frac{\partial^2 v}{\partial t^2} + y_C \frac{\partial^2 w}{\partial t^2} \right) = m_x + p_z y_C - p_y z_C \] (16)

subjected to the following boundary conditions

\[ \alpha_1 v(x,t) + \alpha_2 R_Y(x,t) = \alpha_3 \] (17a)

\[ \alpha_1 \frac{\partial v(x,t)}{\partial x} + \alpha_2 M_Z(x,t) = \alpha_3 \] (17b)

\[ \beta_1 w(x,t) + \beta_2 R_Z(x,t) = \beta_3 \] (18a)

\[ \beta_1 \frac{\partial w(x,t)}{\partial x} + \beta_2 M_Y(x,t) = \beta_3 \] (18b)
\[ \gamma_1 \theta(x,t) + \gamma_2 M_t(x,t) = \gamma_3 \] (19a)
\[ \gamma_1 \frac{\partial \theta(x,t)}{\partial x} + \gamma_2 M_b(x,t) = \gamma_3 \] (19b)

at the beam ends \( x = 0, l \), where \( v = v(x,t) \), \( w = w(x,t) \) are the deflections of the shear center along \( y, z \) axes, respectively; \( \theta(x,t) \) is the angle of twist of the cross-section about the shear center \( S \); \( A \) is the cross-sectional area and \( I_S \) the polar moment of inertia with respect to the shear center \( S \) given as
\[ A = \sum_{j=1}^{K} \rho_j A_j, \] (20a)
\[ I_S = \sum_{j=1}^{K} \rho_j I_{sj} \] (20b)

with \( \rho_j, A_j \ (j = 1, 2, \ldots, K) \) being the mass densities and the areas of the materials, respectively.

The initial conditions are
\[ v(x,0) = \bar{v}_0(x) \] (21a)
\[ \dot{v}(x,0) = \ddot{v}_0(x) \] (21b)
\[ w(x,0) = \bar{w}_0(x) \] (22a)
\[ \dot{w}(x,0) = \ddot{w}_0(x) \] (22b)
\[ \theta(x,0) = \bar{\theta}_0(x) \] (23a)
\[ \dot{\theta}(x,0) = \ddot{\theta}_0(x) \] (23b)

In the boundary conditions (17), (18) \( R_Y, M_Y \) and \( R_Z, M_Z \) are the reactions and bending moments with respect to \( Y \) and \( Z \) axes, respectively, given as
\[ R_Y(x,t) = -E_1 I_Z \frac{\partial^3 v(x,t)}{\partial x^3} - E_1 I_Y \frac{\partial^3 w(x,t)}{\partial x^3} \] (24)
\[ M_Y(x,t) = -E_1 I_Y \frac{\partial^2 w(x,t)}{\partial x^2} - E_1 I_Z \frac{\partial^2 v(x,t)}{\partial x^2} \] (25)
\[ R_Z(x,t) = -E_1 I_Y \frac{\partial^3 w(x,t)}{\partial x^3} - E_1 I_Z \frac{\partial^3 v(x,t)}{\partial x^3} \] (26)
\[ M_Z(x,t) = E_1 I_Z \frac{\partial^2 v(x,t)}{\partial x^2} + E_1 I_Y \frac{\partial^2 w(x,t)}{\partial x^2} \] (27)

while in Eqns. (6) and (19) \( M_t \) and \( M_b \) are the torsional and warping moments, respectively, given as (Sapountzakis and Tsiatas, 2007a) (in the buckling case the time is excluded)
\[ M_t(x,t) = -E_1 C_s \frac{\partial^3 \theta(x,t)}{\partial x^3} + G_1 I_t \frac{\partial \theta(x,t)}{\partial x} \] (28)
\[ M_b(x,t) = -E_1 C_s \frac{\partial^2 \theta(x,t)}{\partial x^2} \] (29)

Finally, \( \alpha_k, \bar{\alpha}_k, \beta_k, \bar{\beta}_k, \gamma_k, \bar{\gamma}_k \) \((k = 1, 2, 3)\) are functions specified at the beam ends \( x = 0, l \). Eqns. (4-6) and (17-19) describe the most general linear boundary conditions associated with the problem at hand and can include elastic support or restrain. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived form these equations by specifying appropriately these functions (e.g. for a clamped edge it is \( \alpha_1 = \beta_1 = \gamma_1 = 1, \bar{\alpha}_1 = \bar{\beta}_1 = \bar{\gamma}_1 = 1, \alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \gamma_2 = \gamma_3 = \bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\beta}_2 = \bar{\beta}_3 = \bar{\gamma}_2 = \bar{\gamma}_3 = 0 \).

3 Integral representations – numerical solution

3.1 The flexural-torsional buckling problem

The flexural-torsional buckling problem of a composite beam reduces in establishing the displacement components \( v(x), w(x) \) and \( \theta(x) \) having continuous derivatives up to the fourth order satisfying the coupled governing Eqns. (1)-(3) inside the beam and the boundary conditions (4)-(6) at the beam ends \( x = 0, l \).

Eqns. (1)-(3) are solved using the Analog Equation Method as it was developed for ordinary differential equations in (Sapountzakis and Katsikadelis, 2000; Katsikadelis and Tsiatas, 2005; Sapountzakis and Tsiatas, 2007b). This method is applied for the problem at hand as follows. Let \( v(x), w(x) \) and \( \theta(x) \) be the sought solution of the boundary value problem described by Eqns. (1)-(3) and (4)-(6). Setting as \( u_1(x) = v(x), u_2(x) = w(x), u_3(x) = \theta(x) \) and differentiating these functions four times yields
\[ \frac{d^4 u_i}{dx^4} = b_i(x) \quad (i = 1, 2, 3) \] (30)
Eqns. (30) indicate that the solution of Eqns. (1)-(3) can be established by solving Eqns. (30) under the same boundary conditions (4)-(6), provided that the fictitious load distributions \( b_i(x) \) \( (i = 1, 2, 3) \) are first established. These distributions can be determined using BEM.

Following the numerical procedure analytically described in (Sapountzakis and Katsikadelis, 2000; Katsikadelis and Tsiatas, 2005) and employing the constant element assumption, the discretized integral form of the solution of Eqns. (30) and their derivatives at the \( N \) collocation points is

\[
\begin{align*}
\hat{u}_j & = C_j \mathbf{b}_j - \left( E_1 \hat{u}_i + E_2 \hat{u}_{i,x} + E_3 \hat{u}_{i,xx} + E_4 \hat{u}_{i,xxx} \right) \\
\hat{u}_{i,x} & = C_2 \mathbf{b}_j - \left( E_1 \hat{u}_i + E_2 \hat{u}_{i,x} + E_3 \hat{u}_{i,xx} \right) \\
\hat{u}_{i,xx} & = C_2 \mathbf{b}_j - \left( E_1 \hat{u}_i + E_2 \hat{u}_{i,xx} + E_3 \hat{u}_{i,xxx} \right) \\
\hat{u}_{i,xxx} & = C_1 \mathbf{b}_j - E_1 \hat{u}_{i,xxx} \\
\end{align*}
\]  

where \( C_j \) \( (j = 1, 2, 3, 4) \) are \( N \times N \) known matrices; \( E_j \) \( (j = 1, 2, 3, 4) \) are \( N \times 2 \) also known matrices and \( \hat{u}_i, \hat{u}_{i,x}, \hat{u}_{i,xx}, \hat{u}_{i,xxx} \) are vectors including the values of \( u_i(x) \) and their derivatives at the \( N \) nodal points. Moreover,

\[
\hat{u}_i = \{ u_i(0) u_i(l) \}^T, \quad (32a)
\]

\[
\hat{u}_{i,x} = \left\{ \frac{d u_i(0)}{dx} \frac{d u_i(l)}{dx} \right\}^T, \quad (32b)
\]

\[
\hat{u}_{i,xx} = \left\{ \frac{d^2 u_i(0)}{dx^2} \frac{d^2 u_i(l)}{dx^2} \right\}^T, \quad (32c)
\]

\[
\hat{u}_{i,xxx} = \left\{ \frac{d^3 u_i(0)}{dx^3} \frac{d^3 u_i(l)}{dx^3} \right\}^T \]

are vectors including the two unknown boundary values of the respective boundary quantities and \( \mathbf{b}_j = \{ b_1 b_2 \ldots b_N \}^T \) \( (i = 1, 2, 3) \) is the vector including the \( N \) unknown nodal values of the fictitious load.

Employing the aforementioned numerical procedure for the coupled boundary conditions (4), (5) the following set of linear equations is obtained

\[
\begin{bmatrix}
D_{11} & D_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & D_{21} & D_{22} & 0 & 0 & 0 & 0 & 0 \\
E_{31} & E_{32} & E_{33} & E_{34} & 0 & 0 & 0 & 0 \\
0 & 0 & D_{42} & E_{43} & E_{44} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & D_{52} & D_{53} & D_{54} & 0 \\
0 & 0 & 0 & 0 & 0 & D_{63} & D_{64} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & D_{74} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{84}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\hat{u}_1 \\
\hat{u}_{1,x} \\
\hat{u}_{1,xx} \\
\hat{u}_{1,xxx} \\
\hat{u}_2 \\
\hat{u}_{2,x} \\
\hat{u}_{2,xx} \\
\hat{u}_{2,xxx}
\end{bmatrix} =
\begin{bmatrix}
\alpha_3 \\
\beta_3 \\
\beta_3 \\
\beta_3 \\
\beta_3 \\
\beta_3 \\
\beta_3 \\
\beta_3
\end{bmatrix} +
\begin{bmatrix}
\alpha_3 \\
\beta_3 \\
\beta_3 \\
\beta_3 \\
\beta_3 \\
\beta_3 \\
\beta_3 \\
\beta_3
\end{bmatrix}
\]

while for the boundary conditions (6) we have

\[
\begin{bmatrix}
E_{11} & E_{12} & 0 & E_{14} \\
0 & E_{22} & E_{23} & 0 \\
E_{31} & E_{32} & E_{33} & E_{34} \\
0 & E_{42} & E_{43} & E_{44}
\end{bmatrix}
\begin{bmatrix}
\hat{u}_3 \\
\hat{u}_{3,x} \\
\hat{u}_{3,xx} \\
\hat{u}_{3,xxx}
\end{bmatrix} =
\begin{bmatrix}
\gamma_3 \\
\gamma_3 \\
\gamma_3 \\
\gamma_3
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

where \( D_{11}, D_{12}, D_{14}, D_{18}, D_{21}, D_{22}, D_{27}, D_{33}, D_{55}, D_{56}, D_{58}, D_{63}, D_{66}, D_{67}, E_{22}, E_{23}, E_{14}, (j = 1, 2, 4) \) are \( 2 \times 2 \) known square matrices including the values of the functions \( a_j, \alpha_j, \beta_j, \beta_j \) \( (j = 1, 2) \) of Eqns. (4)-(6); \( \alpha_3, \alpha_3, \beta_3, \beta_3, \gamma_3, \gamma_3 \) are \( 2 \times 1 \) known column matrices including the boundary values of the functions \( a_3, \alpha_3, \beta_3, \gamma_3, \gamma_3 \) of Eqns. (4)-(6); \( E_{jk}, (j, k = 3, 4) \) are \( 2 \times N \) rectangular known matrices originating from the integration of the kernels on the axis of the beam.

Eqns. (31) after eliminating the boundary quantities employing Eqns. (33) and (34), can be written
as

\[ u_i = T_i b_i + t_i, \quad (35a) \]
\[ u_{ix} = T_{ix} b_i + t_{ix}, \quad (35b) \]
\[ u_{i,xx} = T_{i,xx} b_i + t_{i,xx}, \quad (35c) \]
\[ u_{i,xxx} = T_{i,xxx} b_i + t_{i,xxx}, \quad (35d) \]
\[ u_{i,xxxx} = b_i \quad (35e) \]

where \( T_i, T_{ix}, T_{i,xx}, T_{i,xxx} \) are known \( N \times N \) matrices and \( t_i, t_{ix}, t_{i,xx}, t_{i,xxx} \) are known \( N \times 1 \) matrices. It is worth here noting that for homogeneous boundary conditions \((\alpha_3 = \beta_3 = \gamma_3 = \overline{\gamma}_3 = 0)\) it is \( t_i = t_{ix} = t_{i,xx} = t_{i,xxx} = 0 \).

In the conventional BEM, the load vectors \( b_i \) are known and Eqns. (35) are used to evaluate \( u_i \) and their derivatives at the \( N \) nodal points. This, however, can not be done here since \( b_i \) are unknown. For this purpose, \( 3N \) additional equations are derived, which permit the establishment of \( b_i \). These equations result by applying Eqns. (1)-(3) to the \( N \) collocation points, leading to the formulation of the following set of \( 3N \) simultaneous equations

\[ (A + PB) \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} p_Y \\ p_Z \\ m_x + p_Z y_C - p_Y z_C + G_1 t_{3,xx} \\ t_{1,xx} - z_C t_{3,xx} \\ t_{2,xx} + y_C t_{3,xx} \\ -z_C t_{1,xx} + y_C t_{2,xx} + \hat{I}_{b,3} t_{3,xx} \end{bmatrix} \quad (36) \]

In the above set of equations the matrices \( A \) and \( B \) are evaluated from the expressions

\[ A = \begin{bmatrix} E_1 I_Z & E_1 I_{YZ} & 0 \\ E_1 I_{YZ} & E_1 I_Y & 0 \\ 0 & 0 & E_1 C_S - G_1 I_T_{3,xx} \end{bmatrix} \quad (37) \]
\[ B = \begin{bmatrix} T_{1,xx} & 0 & -z_C T_{3,xx} \\ 0 & T_{2,xx} & y_C T_{3,xx} \\ -z_C T_{1,xx} & y_C T_{2,xx} & \hat{I}_{b,3} T_{3,xx} \end{bmatrix} \quad (38) \]

where \( E_1 I_Y, E_1 I_Z, E_1 I_{YZ}, E_1 C_S, G_1 I_T \) are \( N \times N \) diagonal matrices including the values of the \( E_1 I_Y, E_1 I_Z, E_1 I_{YZ}, E_1 C_S, G_1 I_T \) quantities, respectively, at the \( N \) nodal points. Moreover, \( p_Y, p_Z \) and \( m_x \) are vectors containing the values of the external loading at these points.

Solving the linear system of Eqns. (36) for the fictitious load distributions \( b_1, b_2, b_3 \) the displacements and their derivatives in the interior of the beam are computed using Eqns. (35).

**Buckling equation**

In this case it is \( \alpha_3 = \beta_3 = \overline{\beta}_3 = \gamma_3 = \overline{\gamma}_3 = 0 \) (homogeneous boundary conditions) and \( p_X = p_Y = m_x = 0 \). Thus, Eq. (36) becomes

\[ (A + PB) \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = 0 \quad (39) \]

The condition that Eq. (39) has a non-trivial solution yields the buckling equation

\[ \det(A + PB) = 0 \quad (40) \]

### 3.2 The flexural-torsional vibration problem

The solution procedure to the vibration problem of homogeneous beams has been described in detail in (Sapountzakis and Tsiatas, 2007a). However, for the completeness of the paper we present the semidiscretized equation of motion, which for the case of the composite beam takes the form

\[ M \begin{bmatrix} \ddot{b}_1 \\ \ddot{b}_2 \\ \ddot{b}_3 \end{bmatrix} + K \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = f \quad (41) \]

with

\[ M = \begin{bmatrix} \rho_1 A T_1 & 0 & -\rho_1 A z_C T_3 \\ 0 & \rho_1 A T_2 & \rho_1 A y_C T_3 \\ -\rho_1 A z_C T_1 & \rho_1 A y_C T_2 & \rho_1 I_3 T_3 \end{bmatrix} \quad (42) \]
\[ K = \begin{bmatrix} E_1 I_Z & E_1 I_{YZ} & 0 \\ E_1 I_{YZ} & E_1 I_Y & 0 \\ 0 & 0 & E_1 C_S - G_1 I_T_{3,xx} \end{bmatrix} \quad (43) \]
\[ f = \begin{bmatrix} p_Y \\ p_Z \\ m_x + p_Z y_C - p_Y z_C + G_1 I_T_{3,xx} \end{bmatrix} \quad (44) \]

playing the role of the generalized mass matrix, stiffness matrix and force vector, respectively.

Eq. (41) can be solved numerically, using any time step integration technique, to establish the time dependent vectors \( b_1, b_2, b_3 \). Subsequently, the displacements as well as the stress resultants are computed at any cross-section of the beam.
4 Numerical examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the efficiency, wherever possible the accuracy and the range of applications of the developed method. The program, based on the numerical implementation described in the previous, can be used with no modifications for both homogeneous (Sapountzakis and Tsiatas, 2007a) and composite beams of arbitrary cross section, subjected to any linear boundary conditions and to an arbitrarily dynamic loading.

Example 1

For comparison reasons, the composite beam of Fig. 2, with length \( l = 1.0 \) m, \( v_1 = v_2 = 0.2 \), \( \rho_1 = \rho_2 = 2500 \) kg/m\(^3\), \( b_1 = h_1 = 0.4 \) m has been studied. Three different types starting from a thin-walled and ending with a thick-walled cross-section are considered, that is (i) \( b_2 = h_2 = 0.02 \) m (ii) \( b_2 = h_2 = 0.08 \) m and (iii) \( b_2 = h_2 = 0.20 \) m. In Table 1 the computed values of the buckling load \( P \) for the cases of hinged-hinged (HH), fixed-hinged (FH) and fixed-fixed (FF) boundary conditions are presented. From the obtained results it can be concluded that the influence of the boundary conditions on the buckling load is significant, while the buckling load is increasing monotonically with the ratio \( E_2/E_1 \). Moreover, in Tables 2 through 4 the first seven eigenfrequencies of the aforementioned beam (hinged-hinged boundary conditions) are presented for various values of the ratio \( E_2/E_1 \) as compared with those presented in (Sapountzakis and Tsiatas, 2007a) for \( E_2/E_1 = 1 \) and found to be in excellent agreement.
Table 1: Buckling load $P$ of the composite beam of Example 1.

<table>
<thead>
<tr>
<th>$b_2 = h_2$</th>
<th>0.02m</th>
<th>0.08m</th>
<th>0.20m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_2/E_1$</td>
<td>HH</td>
<td>FF</td>
<td>HH</td>
</tr>
<tr>
<td>0.5</td>
<td>648</td>
<td>776</td>
<td>1008</td>
</tr>
<tr>
<td>1</td>
<td>992</td>
<td>1157</td>
<td>1462</td>
</tr>
<tr>
<td>2</td>
<td>1747</td>
<td>1991</td>
<td>2441</td>
</tr>
</tbody>
</table>

Table 2: Eigenfrequencies of the composite HH beam of Example 1 ($b_2 = h_2 = 0.02$ m).

<table>
<thead>
<tr>
<th>$b_2 = h_2$</th>
<th>$E_2/E_1$</th>
<th>0.02m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>Present study</td>
<td>2.204</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>Present study</td>
<td>5.507</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>Present study (Sapountzakis and Tsiatas, 2007a)</td>
<td>10.407</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>Present study</td>
<td>14.529</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>Present study</td>
<td>17.095</td>
</tr>
<tr>
<td>$\omega_6$</td>
<td>Present study</td>
<td>19.475</td>
</tr>
<tr>
<td>$\omega_7$</td>
<td>Present study</td>
<td>25.639</td>
</tr>
</tbody>
</table>

Table 3: Eigenfrequencies of the composite HH beam of Example 1 ($b_2 = h_2 = 0.08$ m).

<table>
<thead>
<tr>
<th>$b_2 = h_2$</th>
<th>$E_2/E_1$</th>
<th>0.08m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>Present study</td>
<td>7.891</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>Present study</td>
<td>15.812</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>Present study</td>
<td>18.526</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>Present study</td>
<td>21.132</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>Present study</td>
<td>39.552</td>
</tr>
<tr>
<td>$\omega_6$</td>
<td>Present study</td>
<td>56.476</td>
</tr>
<tr>
<td>$\omega_7$</td>
<td>Present study</td>
<td>64.292</td>
</tr>
</tbody>
</table>

Table 4: Eigenfrequencies of the composite HH beam of Example 1 ($b_2 = h_2 = 0.20$ m).

<table>
<thead>
<tr>
<th>$b_2 = h_2$</th>
<th>$E_2/E_1$</th>
<th>0.20m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>Present study</td>
<td>13.351</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>Present study</td>
<td>17.866</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>Present study</td>
<td>24.446</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>Present study</td>
<td>47.682</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>Present study</td>
<td>58.005</td>
</tr>
<tr>
<td>$\omega_6$</td>
<td>Present study</td>
<td>71.478</td>
</tr>
<tr>
<td>$\omega_7$</td>
<td>Present study</td>
<td>81.240</td>
</tr>
</tbody>
</table>
Figure 4: Time history of the displacements for the hinged-hinged beam of Example 2.

Figure 5: Time history of the displacements for the fixed-hinged beam of Example 2.

Figure 6: Time history of the displacements for the fixed-fixed beam of Example 2.
Table 5: Buckling load \( P \) of the composite beam of Example 2.

<table>
<thead>
<tr>
<th>Boundary Conditions</th>
<th>( k_R = 10^5 )</th>
<th>( k_R = 3 \times 10^5 )</th>
<th>( k_R = 5 \times 10^5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>HH</td>
<td>1013656</td>
<td>1750987</td>
<td>1464110</td>
</tr>
<tr>
<td>FH</td>
<td>2121313</td>
<td>2738484</td>
<td>2998312</td>
</tr>
</tbody>
</table>

Table 6: Eigenfrequencies of the composite beam of Example 2.

<table>
<thead>
<tr>
<th>Boundary Conditions</th>
<th>( k_R = 10^7 )</th>
<th>( k_R = 10^8 )</th>
<th>( k_R = 10^9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>HH</td>
<td>3.146</td>
<td>5.640</td>
<td>6.070</td>
</tr>
<tr>
<td>FH</td>
<td>4.500</td>
<td>7.132</td>
<td>7.132</td>
</tr>
<tr>
<td>F-FTS</td>
<td>1013656</td>
<td>1750987</td>
<td>1464110</td>
</tr>
</tbody>
</table>

From the aforementioned tables it can be pointed out that all eigenfrequencies are increasing also monotonically with the ratio \( E_2/E_1 \).

Example 2

The composite beam of length \( l = 40.0 \) m having a cross section consisting of a concrete C20/25 rectangular part stiffened by three concrete C35/45 beams (Fig. 3) has been studied. The data are \( E_1 I_Y = 1.64321 \times 10^8 \) kNm\(^2\), \( E_1 I_Z = 1.17524 \times 10^9 \) kNm\(^2\), \( E_1 C_s = 9.96332 \times 10^8 \) kNm\(^4\), \( G_1 I_t = 4.46380 \times 10^6 \) kNm\(^2\), \( A = 6.40092 \) m\(^2\), \( I_S = 60.38196 \) m\(^4\), \( \rho_1 = \rho_2 = 2.5 \) ton/m\(^3\), \( z_C = 1.48902 \) m. Three types of boundary conditions, namely hinged-hinged (HH), fixed-hinged (FH) and fixed-fixed (FF) are considered, as well as a fixed-fixed with a torsional elastic support (F-FTS) at \( x = l \) (all coefficients in boundary conditions are set to zero except from \( \alpha_1 = \beta_1 = \gamma_1 = \eta_1 = 1 \) at \( x = 0, \) \( l \), \( \gamma_1 = k_R, \) \( \gamma_2 = \eta_1 = 1 \) at \( x = 0 \) and \( \gamma_1 = 1 \) at \( x = l \)). In Tables 5 and 6 the buckling load \( P \) and the first seven eigenfrequencies of the aforementioned beam are presented. Moreover, the forced vibrations arising from the application of the dynamic loading \( p_Z(t) = p_{Z0} \) and \( p_Y(t) = p_{y0} \) with zero initial conditions are examined. In Figs. 4 through 6 the time histories of the displacements at the cross-section \( x = 20.0 \) m, for the three aforementioned types of boundary conditions and \( p_{Z0} = 50 \) kN/m, \( p_{Z0} = 25 \) kN/m are presented. The influence of the boundary conditions on the buckling load, the eigenfrequencies and the response of the beam is pronounced.

5 Conclusions

In this paper a boundary element method is developed for the general flexural-torsional buckling and vibration analysis of composite Euler-Bernoulli beams of arbitrarily shaped cross-section. The composite beam consists of materials in contact each of which can surround a finite number of inclusions. It is subjected to a compressive centrally applied load together with arbitrarily transverse and/or torsional distributed or concentrated loading, while its edges are restrained by the most general linear boundary conditions. The main conclusions that can be drawn from this investigation are:

a. The general character of the proposed method is verified from the fact that all basic equations are formulated with respect to an arbitrary coordinate system, which is not restricted to the principal one.

b. The proposed method can treat composite beams of both thin and thick walled cross sec-
tions taking into account the warping along the thickness of the walls.

c. The developed method overcomes the shortcoming of a possible thin tube theory (TTT) solution, which its utilization has been proven to be prohibitive even in thin walled homogeneous sections.

d. The method permits the inclusion of the torsion-bending and flexural coupling stiffnesses which play an important role in the response of the beam and have to be taken always into account.

e. The influence of the boundary conditions on the buckling load is significant, while the buckling load is increasing monotonically with the ratio $E_2/E_1$.

f. All eigenfrequencies are increasing also monotonically with the ratio $E_2/E_1$.

g. The computer program, based on the numerical implementation described above, can be used with no modifications for both homogeneous and composite beams of arbitrary cross section, subjected to any linear boundary conditions and to an arbitrarily dynamic loading.

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References


