On the Compression Viewed as a Geometric Transformation

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Abstract: A modeling of the compression by using the property of Helmholtz equation to be invariant under geometric transformations is presented in this paper. The versatility of the geometric transformations is illustrated in order to obtain a new interpretation of the compression process. The physical spatial compression leads, most of the times, to new materials with inhomogeneous and anisotropic properties. The compression can be theoretically controlled by the geometric transformations. As an example, new architectures for auxetic materials can be built up by applying the geometric transformations. The new versions are finding their full correspondents in the results of the experiments in which the conventional foams are subjected to compressive cyclic loading.

Keywords: Compression, Geometric transformation, Auxetic foams, Cosserat theory.

1 Introduction

Pendry, Shurig and Smith (2006) proved that a finite size object surrounded by a coating consisting of a specially designed metamaterial would become invisible for electromagnetic waves at any frequency. The idea is that the sound sees the space differently [Dupont et al. (2011)]. For the sound, the concept of distance is modified by the acoustic properties of the regions through which the sound travels. In geometrical acoustics, the idea of the acoustical path when travelling an infinitesimal distance ds, is the corresponding acoustical path length $c^{-1}ds$, where $c^{-1} = \sqrt{\rho/\kappa}$

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with \( \rho \) is the fluid density and \( \kappa \) is the compression modulus of the fluid. Cummer and Schurig (2007) demonstrated that acoustic waves in a fluid undergo the same geometric transformation as electromagnetic waves do and therefore retain their form. For example, the 3D equation for the pressure waves propagating in a bounded fluid region \( \Omega \subset \mathbb{R}^3 \) is the Helmholtz equation

\[
\nabla \cdot (\rho^{-1} \nabla p) + \frac{\omega^2}{\kappa} p = 0,
\]

(1.1)

where \( p \) is the pressure, \( \rho \) is the rank-2 tensor of the fluid density, \( \kappa \) is the compression modulus of the fluid, and \( \omega \) is the wave frequency.

Geometric transformations applied to certain types of elastodynamic waves in structural mechanics received less attention, since the Navier equations do not usually retain their form under geometric changes [Bigoni et al. (1998); Norris (2008)]. For example, the in-plane propagation of time-harmonic elastic waves is governed by the Navier equations

\[
\nabla \cdot C : \nabla U + \rho \omega^2 U + b = 0,
\]

(1.2)

where \( u \) is the displacement, \( \rho \) the density, \( C \) the 4th-order material tensor of the linear elastic material and \( b(x) \) represents the spatial distribution of a simple harmonic body force \( \hat{b}(x,t) = b(x) \exp(i\omega t) \), with the wave-frequency and \( t \) the time.

Brun, Guenneau and Movchan (2009) demonstrated that the Navier equations (1.2) retain their form under the transformation

\[
r' = r_0 + \frac{r - r_0}{r_1}, \quad \theta = \theta'
\]

(1.3)

for \( r \leq r_1 \), \( r = r' \), \( \theta = \theta' \), for \( r > r_1 \), where \( r_0 \) and \( r_1 \) are the inner and outer radii of the circular domain, respectively.

Let us consider the geometric transformation from the coordinate system \((x',y',z')\) of the compressed space to the original coordinate system \((x,y,z)\), given by \( x(x',y',z'), y(x',y',z') \) and \( z(x',y',z') \). The change of coordinates is characterized by the transformation of the differentials through the Jacobian \( J \) of this transformation, i.e.

\[
\begin{pmatrix}
\frac{dx}{d\xi'} \\
\frac{dy}{d\eta'} \\
\frac{dz}{d\zeta'}
\end{pmatrix} = J_{xx'} \begin{pmatrix}
\frac{dx'}{d\xi} \\
\frac{dy'}{d\eta} \\
\frac{dz'}{d\zeta}
\end{pmatrix},
\]

(1.4)

From the geometrical point of view, the change of coordinates implies that, in the transformed region, one can work with an associated metric tensor [Zolla et al. (2007); Guenneau et al. (2011)]

\[
T = \frac{J_{xx'}^T J_{xx}}{\det(J_{xx'})}.
\]

(1.5)
In terms of the material parameters, one can replace the material from the original domain (homogeneous and isotropic) by an equivalent compressed one that is inhomogeneous (its characteristics depend on the spherical \((r', \theta', \phi')\) coordinates) and anisotropic (described by a tensor), and whose properties, in terms of \(J_{x'x}\), are given by

\[
\rho' = J_{x'x}^{-T} \cdot \rho \cdot J_{x'x}^{-1} \cdot \det(J_{x'x}), \quad \kappa' = \kappa \det(J_{x'x}),
\]

or, equivalently, in terms of \(J_{xx'}\)

\[
\rho' = \frac{J_{xx'}^T \cdot \rho \cdot J_{x'x}}{\det(J_{xx'})}, \quad \kappa' = \frac{\kappa}{\det(J_{xx'})}.
\]

(1.6)

Here, \(\rho'\) is a second order tensor. When the Jacobian matrix is diagonal, (1.6) and (1.7) can be more easily written. Multiplying (1.1) by a test function \(\varphi\) and integrating by parts, one obtains [Dupont et al. (2011)]

\[
- \int_{\Omega} (\nabla (x,y,z) \varphi \cdot \rho^{-1} \nabla (x,y,z) p) dV + \int (\omega^2 \kappa^{-1} p \varphi) dV = 0.
\]

(1.8)

In (1.8) the surface integral, corresponding to a Neumann integral over the boundary \(\partial \Omega\), is zero. By applying the coordinate transformation \((x,y,z) \to (x',y',z')\) to (1.8) and using (1.4), one obtains

\[
- \int_{\Omega} (J_{x'x}^T \nabla (x',y',z') \varphi \cdot \rho^{-1} J_{x'x}^T \nabla (x,y,z) p) \det(J_{xx'}) dV' + \int (\det(J_{xx'}) \omega^2 \kappa^{-1} p \varphi) dV' = 0,
\]

(1.9)

in terms of \(J_{xx'}\), and

\[
- \int_{\Omega} \left( (\nabla (x',y',z') \varphi) \cdot \frac{J_{x'x}^T \rho^{-1} J_{x'x}^T \nabla (x',y',z') p}{\det(J_{x'x})} \right) dV' + \int \left( \frac{\kappa^{-1}}{\det(J_{x'x})} \omega^2 p \varphi \right) dV' = 0,
\]

(1.10)

in terms of \(J_{x'x}\).

This theory is applied in this paper to model the compression of the conventional foams. The scope is to transform conventional foam which occupies the disk \(r \leq R_2\) into auxetic material which fills the annulus \(R_1 \leq r \leq R_2\). The new material is inhomogeneous and anisotropic. The problem is related to the current research in invisibility cloaks starting from the works by Pendry, Schurig and Smith (2006); Brun,

Milton, Briane and Willis (2006) showed that geometric transformations cannot be applied to equations which are not invariant under coordinate transformations and, consequently, if cloaking exists for such equations (for example the elasticity equations), it would be of a different nature from acoustic and electromagnetic.

The touchstone of our technique is that the governing equations of the non-auxetic foams are invariant under geometric transformations, more precisely, the equations are reduced to the Helmholtz equations.

2 Formulation of the Problem

The geometric transformation may be linear or nonlinear. A linear geometric transformation (1.4) which maps the disk \( r \leq R_2 \) into an annulus \( R_1 \leq r \leq R_2 \) [Pendry, Schurig and Smith (2006); Chen and Lakes (1989)] is given by

\[
r'(r) = R_1 + r \frac{R_2 - R_1}{R_2}, \quad 0 \leq r \leq R_2,
\]

\[
0 \leq \theta \leq 2\pi,
\]

\[
x'_3 = x_3, \quad x_3 \in \mathbb{R},
\]

where \( r', \theta', x'_3 \) are radially contracted cylindrical coordinates \( r, \theta, x_3 \). The Cartesian basis \((x_1, x_2, x_3)\) is defined as \( x_1 = r \cos \theta, x_2 = r \sin \theta \). The Jacobean of the transformation from polar to stretched polar coordinates is given by \( J \). In the stretched space, the associated metric tensor is given by (1.5)

\[
T = J_{rr'}^T J_{rr} \frac{1}{\det(J_{rr})}.
\]

Qiu et al. (2009) classified the geometric transformation functions in terms of the negative (i.e., concave-down) or positive (i.e., concave-up) sign of the second order derivative of this function. The concave-down nonlinear transformation compresses
a sphere of radius $R_2$ in the original space $\Omega$ into a shell region $R_1 < r' < R_2$ in the compressed space $\Omega'$ as

$$r(\beta) = \frac{R_2^{\beta+1}}{R_2^\beta - R_1^\beta} \left( 1 - \left( \frac{R_1}{r'} \right)^\beta \right),$$

(2.3)

where $\beta$ denotes the degree of the nonlinearity in the transformation. By taking $\beta \to 0$ in (2.3), the linear case is obtained, namely

$$r(\beta) = \frac{R_2 \ln(r'/R_1)}{\ln(R_2/R_1)}.$$  

(2.4)

All curves belonging to (2.3) have negative second order derivative with respect to the physical space $r'$. This class of transformations is termed as the *concave-down* transformation. The transformation function (2.3) depends on the radial component $r'$ in the spherical coordinate system $(r', \theta', \phi')$.

The concave-up nonlinear transformation compresses a sphere of the radius $R_2$ in the original space $\Omega$ into a shell region $R_1 < r' < R_2$ in the compressed space $\Omega'$ as

$$r(\beta) = \frac{R_2 R_1^\beta}{R_2^\beta - R_1^\beta} \left( \left( \frac{r'}{R_1} \right)^\beta - 1 \right).$$

(2.5)

As $\beta \to 0$, one obtains again the linear case (2.4). This class of transformations is termed as the *concave-up* transformation because (2.5) has positive second order derivatives.

Once the above geometric transforms are written, let us formulate the problem to be solved in this paper. Let us suppose that the original domain is a cylinder of radius $R_2$ and length $l$. This domain is filled with conventional non-auxetic cellular foam. The spatial compression is obtained by applying the geometric transformation (2.1). The transformed domain is a shell cylinder of internal and external radii $R_1$ and $R_2$, respectively, and the length $l'$.

As example, this paper is building new architectures for auxetic materials by applying the geometric transformations. These theoretically versions find their full correspondents in the experiments of compression the conventional foams.

In the following we present the basic elements regarding the auxetic materials. The term *auxetic* is coming from the Greek word *auxetos*, meaning *that which may be increased*. Instead of getting thinner like an elongated elastic band, the auxetic material grows fatter, expanding laterally when stretched [Lakes (1986), (1987), (1991)]. Since 1987, when isotropic auxetic foam was manufactured for the first
time, negative Poisson’s ratio materials have created interest for potential engineering applications. A feature that the auxetic materials showed compared to the other foams is the significant damping capacity with increase up to 16 times compared to the conventional foam [Scarpa et al. (2006); Chiroiu et al. (2009); Evans (1991); Donescu, Chiroiu and Munteanu (2009)]. Scientists have known about auxetic materials for over a hundred years, though without giving them much a special attention, and treating them as an accident or a curiosity [Alderson, Alderson and Evans (1997)]. The auxetic behavior is found in materials from the molecular and microscopic levels, up to the macroscopic level. Negative Poisson’s ratios are observed in real materials with a high degree of anisotropy, such as honeycomb structures, reticulated metal foams, re-entrant structures, the skin covering a cow’s teats, certain rocks and minerals, living bone tissue, etc. All the major classes of polymers, composites, metals and ceramics, can exist in the auxetic version [Mihailescu and Chiroiu (2005)].

Cellular solids are two phase composite materials in which one phase is a solid and the other is a fluid, most often air. The positive Poisson’s ratio is the result of the convex shape of cell surfaces. By volume compression, a part of the cell surface may acquire first a zeroth and then negative curvature. When the number of such inverted cells dominates, compressibility of the material rises till the appearance of the negative Poisson’s ratio [Shilko and Konyok (2004); Gaspar et al. (2005)].

Figure 2.1 shows the simplest 2D lattice structures for closed-cell solids: (a) square structure, (b) rectangle structure, (c) regular hexagonal structure, d) triangular structure and d) irregular structure. Figure 2.2 illustrates models of auxetic structures obtained from aforementioned materials by appearance of the negative Poisson’s ratio: a), b) re-entrant honeycomb network, c) Shilko and Konyok model of the inverted closed-cell foam, and d) re-entrant regular array of rectangular nodules interconnected by fibrils [Evans and Alderson (2000)].

The conventional foam exhibits pores with average diameter around 1mm while the auxetic foam has average diameter possible down to a few micrometers or down to a few nanometers.

A comparative analysis between the cyclic loading compressive behaviour of conventional non-auxetic foam and the auxetic foams was performed by Bezazi and Scarpa in 2007. They analyzed the transformation of conventional foam into its auxetic version using a manufacturing process involving compression [Chan and Evans (1997); Scarpa et al. (2004)].

Bezazi and Scarpa (2007) used cylindrical foam samples of 30 mm of diameter and 170 mm of length, and after compression they obtained final diameter of 20 mm and length of 100 mm. The measured density of the auxetic foam was 0.118g / cm$^3$. 
Figure 2.1: Regular 2D cellular solids.

Figure 2.2: Model structures of the inverted closed-cell foam.
The Poisson’s ratio depends on the compressive strain. So, the conventional foam has a Poisson’s ratio positive of 0.25 at compressive strain of 10%, which decreases sharply with the increase of compressive loading, to become slightly negative from 60 to 80% of tensile strain. The auxetic foam exhibits a negative Poisson’s ratio of \(-0.185\) at compressive strain from 10 to 25%, showing a sharp increase for rising compressive strain, reaching then a zero value at 55% of compressive strain and a positive Poisson’s ratio of 1.33 at 80%. The results of the compressive Poisson’s ratio of auxetic foams obtained by Bezazi and Scarpa in 2007, are consistent with the ones presented in the literature [Scarpa et al. (2004); Blake and Isard (1998); Scarpa, Ciffo and Yates (2004); Lakes and Elms (1993)].

3 The Theory of Auxetic Materials

In order to write the equations governing the behavior of the conventional non-auxetic cellular foam, we remind that the microstructure of these foams can be modeled by the micropolar theory.

For this, we suppose that the material is a micropolar solid with chiral effects, i.e. a noncentrosymmetric material. The micropolar and classical theories of elasticity are continuum theories, which make no reference to atoms or other structural features of the material, which is described. Elasticity theory represents more than an analytical description of the phenomenological behavior since it can be derived as a first approximation of the interaction between atoms in the solid [Cosserat (1909); Kröner (1963)]. Interatomic forces are of short range; but they exert more influence than one atomic space. The characteristic length \(l\) in this case should be on the order of the atomic spacing. Phenomena associated with micropolar elasticity are likely to be of larger magnitude, and therefore of greater interest in materials such as cellular solids with larger scale structural features. In fibrous composites, the characteristic length \(l\) may be of the order of the spacing between fibers [Hlavacek (1975), (1976)], in cellular solids it may be comparable to the average cell size [Adomeit (1967); Berglund (1982)].

Consider a chiral Cosserat medium medium, in a Cartesian coordinates system \((x,y,z)\). The equations of motion in the absence of body forces and body couples are [Eringen (1966), (1968); Mindlin (1964), (1965); Chiroiu, Munteanu and Gliozzi (2010)]

\[
\sigma_{kl,k} - \rho \dot{u}_l = 0, \quad m_{rk,r} + \varepsilon_{klr} \sigma_{lr} - \rho j \phi_k = 0, \quad k, l = 1, 2, 3. \tag{3.1}
\]

Here \(\sigma_{kl}\) is the stress tensor, \(m_{kl}\) is the couple stress tensor, \(u\) is the displacement vector, \(\phi_k\) is the microrotation vector which in Cosserat elasticity is kinematically different from the macrorotation vector \(r_k = \frac{1}{2} \varepsilon_{klm} u_{m,l}\), and \(\varepsilon_{klm}\) is the Levi-Civita
permutation symbol. We remember that $\varphi_k$ refers to the rotation of points themselves, while $r_k$ refers to the rotation associated with movement of nearby points. In (3.1) $\rho$ is the mass density and $f$ the microinertia. The constitutive equations are given by [Donescu, Chiroiu and Munteanu (2009)]

$$
\sigma_{kl} = \lambda e_{rr}\delta_{kl} + (2\mu + \kappa)e_{kl} + \kappa\epsilon_{klm}(r_m - \varphi_m) + C_1\varphi_{r,r}\delta_{kl} + C_2\varphi_{l,l} + C_3\varphi_{l,k},
$$

$$
m_{kl} = \alpha\varphi_{r,r}\delta_{kl} + \beta\varphi_{k,l} + \gamma\varphi_{l,k} + C_1e_{rr}\delta_{kl} + (C_2 + C_3)e_{kl} + (C_3 - C_2)\epsilon_{klm}(r_m - \varphi_m),
$$

(3.2)

where $e_{kl} = (u_{k,l} + u_{l,k})/2$ is the macrostrain vector. $\lambda$, and $\mu$ are Lamé elastic constants, $\kappa$ is the Cosserat rotation modulus and $\alpha, \beta, \gamma$ the Cosserat rotation gradient moduli, and $C_i, i = 1, 2, 3$, the chiral elastic constants associated with noncentrosymmetry [Cosserat (1909); Kröner (1963); Lakes and Benedict (1982); Lakes (2001)]. The materials which are not invariant to coordinates inversions can have a qualitatively different behavior in comparison with isotropic solids. An elastic chiral material (noncentrosymmetric material) is isotropic with respect to coordinate rotations but not with respect to inversions. For $C_i = 0$ the equations of isotropic micropolar elasticity are recovered. For $\alpha = \beta = \gamma = \kappa = 0$, (3.4) reduces to the constitutive equations of classical isotropic linear elasticity theory. From the requirement that the internal energy must be nonnegative (the material is stable), we obtain restrictions on the elastic constants $0 \leq 3\lambda + 2\mu + \kappa$, $0 \leq 2\mu + \kappa$, $0 \leq \kappa$, $-\gamma \leq \beta \leq \gamma$, $0 \leq \gamma$ and any positive or negative $C_1, C_2, C_3$ [Gauthier (1982); Lakes and Benedict (1982)].

By definition, the set composed of the asymmetric tensors $\sigma_{kl}, m_{kl}, e_{kl}, k, l = 1, 2, 3$, and the vectors $u_k, \varphi_k$, namely $F = \{\sigma_{kl}, m_{kl}, u_k, \varphi_k, \ k, l = 1, 2, 3\}$ is said to be an elastodynamic state on the bounded medium, if it satisfies (3.1) and (3.2). It is easy to prove that the state $F$ can be decomposed in the form

$$
F = F_1 + F_2,
$$

(3.3)

where $F_1 = \{\sigma_{11}, \sigma_{13}, \sigma_{33}, m_{22}, u_1, u_3, \varphi_2\}$ and $F_2 = \{\sigma_{22}, m_{11}, m_{13}, m_{33}, u_2, \varphi_1, \varphi_3\}$ [Donescu, Chiroiu and Munteanu (2009)]. By introducing (3.2) into (3.1), after a proper manipulation of equations, the following equations in the unknowns $u = (u_1, u_2, u_3)$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ are found.

$$
(\lambda + 2\mu + \kappa)\nabla\nabla u - (\mu + \kappa)K_0^2\nabla \times \nabla \times u + \kappa(1 - K_0^2)\nabla \times \varphi = \rho \ddot{u},
$$

(3.4)

$$
(\alpha + \beta + \gamma)\nabla\nabla \varphi - \gamma K_0^2\nabla \times \nabla \times \varphi + \kappa(1 - K_0^2)\nabla \times u - 2\kappa(1 - K_0^2)\varphi = \rho \dot{j}\varphi,
$$

(3.5)
where

\[ K_0^2 = 1 + \frac{(C_1 + C_2 + C_3)^2}{(\lambda + 2\mu + \kappa)(\alpha + \beta + \gamma)}. \]

We see that (3.4) and (3.5) are decoupled into two sets of equations with respect to \( F_1, F_2 \), respectively. For simplicity, without any loss of generality, the particular case in which all quantities depend only on \( x \) (radial direction \( r \)) and \( y \) (longitudinal direction) is considered. We will focus on the 2D set of equations corresponding to \( F_1 \), the other set being solved in a similar manner. By introducing the dimensionless quantities

\[
\begin{align*}
    x' &= \frac{\omega}{c_1} x, \quad y' = \frac{\omega}{c_1} y, \quad v'_i = \frac{\omega}{c_1} u_i, \quad i = 1, 2, \\
    \phi'_2 &= \frac{\mu K_0^2}{\rho j \omega^2} \phi_2, \quad t' = \omega t, \quad \sigma'_{ij} = \frac{1}{\mu K_0^2} \sigma_{ij}, \\
    m'_{ij} &= \frac{c_1}{\gamma \omega K_0^2} m_{ij}, \quad \omega^2 = \frac{\kappa(1 - K_0^2)}{\rho j}, \quad c_1^2 = \frac{\lambda + 2\mu + \kappa}{\rho}, \quad i, j = 1, 3,
\end{align*}
\]

the equations (3.4) and (3.5) reduce to by suppressing the dashes

\[
\begin{align*}
    (s_1 + s_2) v_{1,xx} + s_1 v_{3,xy} + s_2 v_{1,yy} - s_3 \phi_{2,y} + \omega^2 v_1 &= 0, \\
    (s_1 + s_2) v_{3,yy} + s_1 v_{1,xy} + s_2 v_{3,xx} + s_3 \phi_{2,x} + \omega^2 v_3 &= 0, \\
    \phi_{2,xx} + \phi_{2,yy} - s_5 \phi_2 + s_6 (v_{1,y} - v_{3,x}) + s_4 \omega^2 \phi_2 &= 0,
\end{align*}
\]

where

\[
\begin{align*}
    s_1 &= \frac{\lambda + \mu K_0^2}{\rho c_1^2}, \quad s_2 = \frac{\kappa(1 - K_0^2) + \mu K_0^2}{\rho c_1^2}, \quad s_3 = \frac{\kappa j(1 - K_0^2) \omega^2}{\mu K_0^2 c_1^2}, \\
    s_4 &= \frac{\rho j c_1^2}{\gamma K_0^2}, \quad s_5 = \frac{2c_1^2 \kappa(1 - K_0^2)}{\omega^2 \gamma K_0^2}, \quad s_6 = \frac{c_1^2 \mu}{\omega^2 \gamma}.
\end{align*}
\]

The equations (3.6) can be rewritten under the form

\[
\nabla \cdot S : \nabla U + \omega^2 U = 0,
\]

where \( \rho \) is the scalar density of the elastic medium, \( S \) is the fourth-order material tensor, \( \omega \) is the wave angular frequency, and \( U(x_1, x_2, x_3, t) = U(x_1, x_2, x_3) \exp(-i\omega t) \) is the vector defined as \( U = (v_1, v_3, \phi_2) \). The equations (3.8) retain their form under this transforms (2.1), (2.3) and (2.5).
Under a change of coordinates \((x', y', z')\) to \((x, y, z)\) given by (3.1) such that \(U'(x') = J_{x'}^{-1}U(x)\), (eq. (3.8) takes the form

\[
\nabla' \cdot S' : \nabla' U' + \omega'^2 U' = 0,
\]

(3.9)

with \(U' = (v'_1, v'_3, \phi'_2)\). Eqs. (3.9) can be written explicitly as

\[
(s'_1 + s'_2)v'_{1,x'} + s'_1v'_{3,y'} + s'_2v_{1,y'} - s'_3\phi'_{2,y'} + \omega'^2v'_1 = 0,
\]

(3.10)

\[
(s'_1 + s'_2)v'_{3,y'} + s'_1v'_{1,x'} + s'_2v'_{3,x'} + s'_3\phi'_{2,x'} + \omega'^2v'_3 = 0,
\]

\[
\phi'_{2,x'} + \phi'_{2,y'} + s'_5\phi'_2 + s'_6(\nu'_{1,y'} - \nu'_{3,x'}) + s'_4\omega'^2\phi'_2 = 0,
\]

with

\[
s'_1 = \frac{\lambda' + \mu'K'_{0}}{\rho'c'^2}, \quad s'_2 = \frac{\kappa'(1 - K'^2_{0}) + \mu'K'^2_{0}}{\rho'c'^2}, \quad s'_3 = \frac{\kappa'(1 - K'^2_{0})\omega'^2}{\mu'K'^2_{0}c'^2},
\]

\[
s'_4 = \frac{\gamma'K'_{0}}{\rho'c'^2}, \quad s'_5 = \frac{2c'^2\kappa'(1 - K'^2_{0})}{\omega'^2\gamma'K'^2_{0}}, \quad s'_6 = \frac{c'^2\mu'}{\omega'^2\gamma'},
\]

(3.11)

\[K'^2_{0} = 1 + \frac{(c'_{1} + c'_{2} + c'_{3})^2}{(\lambda' + 2\mu' + \kappa')(\alpha' + \beta' + \gamma')},\]

where

\[
\lambda' = \frac{\lambda}{\det(J_{x'})}, \quad \mu' = \frac{\mu}{\det(J_{x'})}, \quad \kappa' = \frac{\kappa}{\det(J_{x'})},
\]

\[
\alpha' = \frac{\alpha}{\det(J_{x'})}, \quad \beta' = \frac{\beta}{\det(J_{x'})}, \quad \gamma' = \frac{\gamma}{\det(J_{x'})},
\]

\[
\rho' = \frac{\rho}{\det(J_{x'})}, \quad \kappa' = \frac{\kappa}{\det(J_{x'})}, \quad \rho' = \frac{\rho}{\det(J_{x'})}, \quad c'_i = \frac{C_i}{\det(J_{x'})}, \quad i = 1, 2, 3.
\]

(3.12)

When we change the coordinate system, automatically we replace the initial material properties by equivalent material properties given by (3.11) and (3.12).
4 Results

The original material is conventional closed-cell polyurethane foam with \( \rho_{\text{conv}} = 27\text{kg/m}^3 \) density and Poisson’s ratio at tensile test \( \nu_{\text{tens}} = 0.47 \) and the compressive test \( \nu_{\text{comp}} = 0.27 \). The cylindrical specimen has \( R_2 = 15\text{mm} \) initial radius and \( l = 170\text{mm} \) initial length. The following constants are considered: \( \lambda = 2.59\text{GPa}, \mu = 0.77\text{GPa}, \kappa = 0.0144\text{GPa}, \alpha = 1.77 \times 10^4\text{N}, \beta = 3.37 \times 10^4\text{N}, \gamma = 0.33 \times 10^4\text{N}, C_1 = -0.5 \times 10^4\text{N/m}, C_2 = -2.9 \times 10^4\text{N/m}, C_3 = -6.8 \times 10^4\text{N/m}, j = 2 \times 10^{-7}\text{m}^2 \).

We must say that the condition of a positive Young’s modulus and \(-1 < \nu < 0.5\) corresponds to the usual range of properties for stability of the material. The existence of negative material constants (shear modulus, bulk modulus, stiffness) is also permitted (experimentally reported in Wang and Lakes (2004), Teodorescu et al. (2005a,b)).

The Poisson’s ratio \( \nu = \nu_{yx} \) (for tensile and compressive tests) was calculated as the negative ratio between the radial and longitudinal strains using a best fit to the strain-strain graph

\[
\nu_{yx} = -\frac{\varepsilon_x}{\varepsilon_y}. \tag{4.1}
\]

The most important physical parameter to dominate the negative Poisson’s ratio transformation is the compression ratio \( \vartheta = \frac{(R_2^\prime - R_1^\prime)l^\prime}{R_2^\prime l} \), where prime denotes the final parameters.

The existing methods concerning the solving of (3.10) are mainly based on the homogenization approach [Kumar and McDowell (2004)]. Another method is the missing ribs method by Smith et al. (2000). We solve the equations (3.12) by applying the Laplace and Fourier transforms.

Figure 4.1 shows the variation of the Poisson’s ratio with respect to \( 1 - \vartheta \) (equivalent to the compressive strain) for conventional foam (the upper curve) and auxetic foam (the lower curve) respectively. We observe that the conventional foam becomes auxetic \((-0.15 < \nu < 0)\) for \( 0.55 < 1 - \vartheta < 0.77 \), or \( 0.23 < \vartheta < 0.45 \). It is very interesting to see that the auxetic foam is changing the sign for its Poisson’s ratio for \( 0.46 > 1 - \vartheta \). It is of interest to underline that the results provides an overall agreement with the experimental values for the auxetic foam [Bezazi and Scarpa (2007); Shilko and Konyok (2004)].

The initial domain with \( R_2 = 15\text{mm} \) and \( l = 170\text{mm} \) is transformed into a shell cylinder with \( l^\prime = 100\text{mm} \), \( R_2^\prime = 15\text{mm} \) and \( 14.415\text{mm} < R_1^\prime < 14.7\text{mm} \). The transformed annulus domains are presented in Fig. 4.2, for \( \vartheta = 0.25, 0.26, 0.3 \) and 0.4.
Let us introduce three reduced moduli defined as

\[
K'_1 = \frac{K'^2_{10}}{K'^2_0}, \quad K'_2 = \frac{K'^2_{11}}{K'^2_0}, \quad K'_3 = \frac{K'^2_{12}}{K'^2_0},
\]

where \( K'^2_{10}, K'^2_{11} \) and \( K'^2_{12} \) are given by

\[
K'^2_{10} = \frac{(c'_2 + c'_3)^2}{4(2\mu' + \kappa')(\beta' + \gamma')}, \quad K'^2_{11} = \frac{(c'_2 - c'_3)^2}{4(2\mu' + \kappa')(\gamma' - \beta')},
\]

\[
K'^2_{12} = \frac{(3c'_1 + c'_2 + c'_3)^2}{4(3\lambda' + 2\mu' + \kappa')(3\alpha' + \beta' + \gamma')},
\]

and \( K'^2_0 \) is given by (3.11)_3.

The dependence of the properties of the auxetic foam on the Poisson’s ratio \( \nu \) and the coordinates is illustrates next. The variation of the reduced moduli on \( \nu \) is displayed in Figure 4.3. For a given \( \nu \) it is possible to determine a set of permissible material constants, negative or not.

Each set of constants may represent a certain possible structure with demonstrable auxetic properties. The variation of the Young’s modulus with respect to radial...
Figure 4.2: Transformed domains.

Figure 4.3: Reduced moduli versus Poisson’s ratio.
Figure 4.4: Variation of the Young’s modulus with respect to radial coordinate.

Figure 4.5: Variation of the chiral elastic constants $C_1, C_2, C_3$ with respect to radial coordinate.
coordinate $R_1 \leq r \leq R_2$ is presented in Figure 4.4 for $\varphi = 0.25, 0.26, 0.3$ and 0.4 (the corresponding thicknesses for the annulus $R_1 \leq r \leq R_2$ are 0.322mm, 0.34mm, 0.3888 and 0.519 respectively).

The variation of the chiral elastic constants $C_1, C_2$ and $C_3$ with respect to radial coordinate $r$ is illustrated in Figure 4.5, for $\varphi = 0.26$ and 0.4. We see that these constants are negative.

5 Conclusions

A new technique for transforming the conventional foams into auxetic foams is proposed in this paper by exploiting the property of the governing equations to be written in a covariant form such that the metric is only involved in the material parameters. The geometric transformations lead to material properties that are, if not impossible to obtain, at least challenging for manufacture of new materials.

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