Investigation on the Normal Derivative Equation of Helmholtz Integral Equation in Acoustics

Zai You Yan\textsuperscript{1,2}, Fang Sen Cui\textsuperscript{2}, and Kin Chew Hung\textsuperscript{2}

Abstract: Taking the normal derivative of solid angles on the surface into account, a modified Burton and Miller’s formulation is derived. From which, a more reasonable expression of the hypersingular operator is obtained. To overcome the hypersingular integral, the regularization scheme developed recently is employed. Plane acoustic wave scattering from a rigid sphere is computed to show the correctness of the modified formulation with the regularization scheme. In the computation, eight-noded isoparametric element is applied.

Keyword: Boundary element method, solid angle, hypersingular integral, Helmholtz integral equation, regularization

1 Introduction

Boundary element method for acoustic problems has been studied for many years. It is well known that the classical boundary element method for exterior acoustic problems fails to provide a unique solution at certain frequencies, which are the characteristics of the associated interior problem. The nonuniqueness is a purely mathematical problem arising from the boundary integral formulation rather than from the nature of the physical problem. Detailed description about the nonuniqueness problem is presented by Schenck (1968), Burton and Miller (1971), Cisowski and Brebbia (1991), Bentheim and Scheck (1996). The composite Helmholtz integral equation (CHIE) presented by Burton and Miller (1971) is one of the most popular approaches to overcome the well-known nonuniqueness problem. It involves a linear combination of the surface Helmholtz integral equation (HIE) and its normal derivative equation. It is a combined use of dual integral representation for acoustic problem

\[ \text{Chen, Hong (1999).} \] Burton and Miller proved that CHIE would yield a unique solution for all frequencies if a suitable complex multiplicative constant was chosen, even if both the HIE and its normal derivative equation suffered from the nonuniqueness problem. This method appears to be more robust for numerical implementation. However, it still suffers from a major drawback that a hypersingular integral is involved in the normal derivative equation of the HIE.


\begin{footnotesize}
\item[1] Shama Tehchnologies (S) Pte Ltd, Blk 67 Ayer Rajah Crescent #03-01/02, Ayer Rajah Industrial Estate, Singapore 139950
\item[2] Institute of High Performance Computing, 1 Science Park Road, #01-01 The Capricorn Singapore Science Park II, Singapore 117528
\end{footnotesize}
agreed well with the corresponding analytical solutions. Taking the normal derivative of the solid angles on the surface into consideration, Wu (2000) derived a new formulation of the normal derivative of the surface Helmholtz integral equation. He termed it as a global approach for the normal derivative integral equation. No discussion was presented about the expression of the hypersingular operator $N_d$. Also taking the normal derivative of the solid angles on the surface into account, Liu, Cai, Zhao, Zheng and Lam (2002) developed an improved Burton and Miller’s formulation. However, in their derivation, they still expressed the hypersingular operator $N_d$ by taking the operator $\partial/\partial n_p$ inside the integral directly. Therefore, their final equation is not complete.

Recently, several novel boundary integral equations are derived and implemented to avoid the hypersingular integral. Qian, Han, Atluri (2004) [see also Han & Atluri (2003a,b)] derived the symmetric Galerkin boundary element formulations of the regularized forms of non-hypersingular boundary integral equations. Meantime, Qian, Han, Ufimtsev, Atluri (2004) presented the non-hypersingular boundary integral equations by the collocation based boundary element method. Yang (2004) presented the numerical implementation for a boundary integral equation method proposed by Krutitskii (2000).

In this paper, a modified composite Helmholtz integral equation (MCHIE) is presented. The MCHIE takes into account the normal derivative of the solid angles on the surface. We find that the MCHIE is the same as that derived by Chien, Rajiyah and Atluri (1990), Hwang (1997). However, they applied the assumption that the normal derivative of the solid angles on the surface is zero. While in the MCHIE, the normal derivative of solid angles on the surface is taken into consideration. At the same time, a more reasonable explicit expression of the hypersingular operator $N_d$ is derived. The new expression for $N_d$ will overcome the confusion that there are two different expressions for one operator. An improved regularization relationship developed by Yan (2000), Yan Z.Y., Jiang, He and Yan M. (2001), Yan, Hung and Zheng (2003) is successfully applied to overcome the hypersingular integrals in the final equation. Plane acoustic wave scattering from rigid sphere is presented to show the correctness of the MCHIE and the regularization scheme. The numerical results obtained using MCHIE agree well with corresponding analytical solutions.

2 Theoretical formulations

Consider the acoustic problem in the exterior domain. The acoustic field is either radiated by a vibrating object or scattered from a rigid structure. Here we assume that the structure has smooth surface. The governing equation for the acoustic pressure $\phi(p)$ is the Helmholtz integral equation.

$$c(p)\phi(p) = \iint_S \left[ \phi(q) \frac{\partial G_k(p,q)}{\partial n_q} - G_k(p,q) \frac{\partial \phi(q)}{\partial n_q} \right] dS_q$$

(1)

The free-space Green’s function $G_k$ for three-dimensional acoustic problem is given by,

$$G_k(p,q) = e^{-ikr} / 4\pi r \quad r = |p - q|$$

(2)

where $p$ and $q$ are respectively the source point and the field point on the structure surface. $k$ is the wave number. $\vec{n}$ represents the inward normal vector, as shown in Fig. 1. $c(p)$ represents the dimensionless solid angle [Ciskowski and Brebbia (1991)] at point $p$.

$$c(p) = 1 + \iint_S \frac{\partial G_0(p,q)}{\partial n_q} dS_q$$

(3)

and

$$G_0(p,q) = 1 / 4\pi r$$

(4)

\[\begin{array}{c}
\text{E} \\
\vec{n} \\
\text{S}
\end{array}\]

\[\begin{array}{c}
\text{c(p) = 1} \\
\text{c(p) = 1/2} \\
\text{c(p) = 0}
\end{array}\]

\[\begin{array}{l}
p \text{ in the exterior domain} \\
p \text{ on the surfaces} \\
p \text{ in the interior domain}
\end{array}\]

Figure 1: Dimensionless solid angles along the normal direction

For problems considered currently, we have

$$c(p) = \begin{cases} 
1 & p \text{ in the exterior domain} \\
1/2 & p \text{ on the surfaces} \\
0 & p \text{ in the interior domain}
\end{cases}$$

(5)
Investigation on the Normal Derivative Equation

See Fig. 1, the left and right limits of the normal derivative of the solid angle can be expressed as Eq. (6) and (7). Obviously, the normal derivative of the solid angle does not equal to zero. Strictly speaking, no derivative exits if the function is discontinuous. However for some special functions, general derivatives in some sense exist even if the functions are discontinuous. For example, the Heaviside step function.

\[
\lim_{\varepsilon \to 0^+} \frac{\partial c(p)}{\partial n} = \frac{1}{2} - \frac{0}{\varepsilon} = \frac{1}{2\varepsilon}
\]

(6)

\[
\lim_{\varepsilon \to 0^-} \frac{\partial c(p)}{\partial n} = 1 - \frac{1/2}{\varepsilon} = \frac{1}{2\varepsilon}
\]

(7)

In operator notation [Mathews (1986)], Eq. (1) can be written as

\[[-c(p)I + M_k]\varphi = L_k \frac{\partial \varphi}{\partial n}
\]

(8)

where the integral operators \(L_k\) and \(M_k\) are defined as

\[L_k \mu = \int_S \mu(q) G_k(p,q) dS_q
\]

(9)

\[M_k \mu = \int_S \mu(q) \frac{\partial G_k(p,q)}{\partial n_q} dS_q
\]

(10)

It is well known that the classical Helmholtz integral equation Eq. (1) for exterior acoustic problems fails to provide a unique solution at certain characteristic frequencies [Schenck (1968), Burton and Miller (1971), Benthien and Scheck (1996)]. To overcome the nonuniqueness problem, Burton and Miller (1971) developed the composite Helmholtz integral equation (CHIE), which consists of a linear combination of the surface Helmholtz integral equation and its normal derivative equation. In operator notation [Mathews (1986)], it can be expressed as

\[\left\{-\frac{1}{2}I + M_k + \alpha N_d\right\} \varphi = \left\{L_k + \alpha \left[\frac{1}{2}I + M_k^T\right]\right\} \frac{\partial \varphi}{\partial n}
\]

(11)

where \(\alpha\) usually takes the value \(-i/k\). The integral operators \(M_k^T\) and \(N_d\) can be expressed as

\[M_k^T \mu = \int_S \mu(q) \frac{\partial G_k(p,q)}{\partial n_p} dS_q
\]

(12)

\[N_d \mu = \frac{\partial}{\partial n_p} \int_S \mu(q) \frac{\partial G_k(p,q)}{\partial n_q} dS_q
\]

(13)

To derive Eq. (11), Burton and Miller (1971) directly took the normal derivative of the surface Helmholtz equation, which was derived from Eq. (1) as the dimensionless solid angle \(c(p)\) equaled to 1/2. As a result, they had

\[\frac{1}{2} \frac{\partial \varphi(p)}{\partial n_p} = \frac{\partial}{\partial n_p} \int_S \left[\varphi(q) \frac{\partial G_k(p,q)}{\partial n_q} - G_k(p,q) \frac{\partial \varphi(q)}{\partial n_q}\right] dS_q
\]

(14)

In Eq. (1), they applied the condition that the dimensionless solid angles \(c(p)\) on surface were constant and equaled to 1/2. Therefore, they implicitly applied the assumption that on the surface \(\partial c(p)/\partial n_p\) equaled to zero. However, Eq. (6) and (7) have shown that \(\partial c(p)/\partial n_p\) does not equal to zero even though the dimensionless solid angles \(c(p)\) are constant on the surface.

Now, the normal derivative of the dimensionless solid angles will be taken into account to derive the normal derivative equation of the surface Helmholtz integral equation. The derivation is a little different from that presented by Wu (2000).

Taking the normal derivative of Eq. (1) at a point \(p\) on the surface,

\[\frac{\partial}{\partial n_p} [c(p) \varphi(p)] = \frac{\partial}{\partial n_p} \int_S \left[\varphi(q) \frac{\partial G_k(p,q)}{\partial n_q} - G_k(p,q) \frac{\partial \varphi(q)}{\partial n_q}\right] dS_q
\]

(15)

Eq. (15) can be rewritten as

\[c(p) \frac{\partial \varphi(p)}{\partial n_p} = -\int_S \frac{\partial G_k(p,q)}{\partial n_p} \frac{\partial \varphi(q)}{\partial n_q} dS_q + \frac{\partial}{\partial n_p} \int_S \frac{\partial G_k(p,q)}{\partial n_q} \varphi(q) dS_q - \frac{\partial c(p)}{\partial n_p} \varphi(p)
\]

(16)

Compared to the corresponding Eq. (1) in the Burton and Miller’s formulation (1971), an additional term \(-\varphi(p) \partial c(p)/\partial n_p\) appears in the right hand side. Substituting Eq. (3) into the third term on the right hand
side, Eq. (16) can be rewritten as

\[ c(p) \frac{\partial \phi(p)}{\partial n_p} = - \iint_S \frac{\partial G_k(p,q)}{\partial n_p} \frac{\partial \phi(q)}{\partial n_q} dS_q + \frac{\partial}{\partial n_p} \left[ \int_S \frac{\partial G_k(p,q)}{\partial n_q} \phi(q) dS_q \right] + \frac{\partial}{\partial n_p} \left[ \int_S \frac{\partial G_0(p,q)}{\partial n_q} \phi(p) dS_q \right] - \frac{\partial \phi(p)}{\partial n_p} \iint_S \frac{\partial G_0(p,q)}{\partial n_q} dS_q \]

(17)

Then, to ensure the non-integrable singularity in second term on the right hand side of Eq. (17) be integrable, some operation is applied. That is,

\[ c(p) \frac{\partial \phi(p)}{\partial n_p} = - \iint_S \frac{\partial G_k(p,q)}{\partial n_p} \frac{\partial \phi(q)}{\partial n_q} dS_q + \frac{\partial}{\partial n_p} \left[ \int_S \frac{\partial G_k(p,q)}{\partial n_q} \phi(q) dS_q \right]
+ \frac{\partial}{\partial n_p} \left[ \int_S \frac{\partial G_0(p,q)}{\partial n_q} \phi(p) dS_q \right] - \frac{\partial \phi(p)}{\partial n_p} \iint_S \frac{\partial G_0(p,q)}{\partial n_q} dS_q \]

(18)

Therefore

\[ c(p) \frac{\partial \phi(p)}{\partial n_p} = - \iint_S \frac{\partial G_k(p,q)}{\partial n_p} \frac{\partial \phi(q)}{\partial n_q} dS_q + \frac{\partial}{\partial n_p} \left[ \int_S \frac{\partial^2 G_k(p,q)}{\partial n_q \partial n_q} \phi(q) dS_q \right]
+ \frac{\partial}{\partial n_p} \left[ \int_S \frac{\partial^2 G_0(p,q)}{\partial n_q \partial n_q} \phi(p) dS_q \right] - \frac{\partial \phi(p)}{\partial n_p} \iint_S \frac{\partial^2 G_0(p,q)}{\partial n_q \partial n_q} dS_q \]

(19)

Because

\[ \frac{\partial}{\partial n_p} \left[ \int_S \frac{\partial G_0(p,q)}{\partial n_q} \phi(p) dS_q \right] = \phi(p) \frac{\partial}{\partial n_p} \left[ \int_S \frac{\partial G_0(p,q)}{\partial n_q} dS_q \right] + \int_S \left[ \frac{\partial^2 G_0(p,q)}{\partial n_p \partial n_q} \phi(p) \right] dS_q \]

(20)

Eq. (19) can be re-written as

\[ c(p) \frac{\partial \phi(p)}{\partial n_p} = - \iint_S \frac{\partial G_k(p,q)}{\partial n_p} \frac{\partial \phi(q)}{\partial n_q} dS_q + \frac{\partial}{\partial n_p} \left[ \int_S \frac{\partial^2 G_k(p,q)}{\partial n_q \partial n_q} \phi(q) dS_q \right]
+ \frac{\partial}{\partial n_p} \left[ \int_S \frac{\partial^2 G_0(p,q)}{\partial n_q \partial n_q} \phi(p) dS_q \right] - \frac{\partial \phi(p)}{\partial n_p} \iint_S \frac{\partial^2 G_0(p,q)}{\partial n_q \partial n_q} dS_q \]

(21)

Furthermore,

\[ c(p) \frac{\partial \phi(p)}{\partial n_p} = - \iint_S \frac{\partial G_k(p,q)}{\partial n_p} \frac{\partial \phi(q)}{\partial n_q} dS_q + \frac{\partial}{\partial n_p} \left[ \int_S \frac{\partial^2 G_k(p,q)}{\partial n_q \partial n_q} \phi(q) dS_q \right]
+ \frac{\partial}{\partial n_p} \left[ \int_S \frac{\partial^2 G_0(p,q)}{\partial n_q \partial n_q} \phi(p) dS_q \right] - \frac{\partial \phi(p)}{\partial n_p} \iint_S \frac{\partial^2 G_0(p,q)}{\partial n_q \partial n_q} dS_q \]

(22)

Then a modified Burton and Miller’s formulation or a modified composite Helmholtz integral equation (MCHIE) is developed by a linear combination of Eq. (22) and Eq. (1) on the surface. In operator notation, it is

\[ \left\{ - \frac{1}{2} I + M_k + \alpha [ (N_k - N_0) + N_0 - N_0 - I ] \right\} \phi = \left\{ L_k + \alpha \left[ \frac{1}{2} I + M_k^I \right] \right\} \frac{\partial \phi}{\partial n} \]

(23)

where \( N_0 \) is a function of point \( p \) rather than an integral operator,

\[ N_0 = \iint_S \frac{\partial^2 G_0(p,q)}{\partial n_p \partial n_q} dS_q \]

(24)

An improved regularization relationship developed by Yan (2000), Yan Z.Y., Jiang, He and Yan M. (2001), Yan,
Hung and Zheng (2003) can be applied in the numerical computation of Eq. (23). Because Eq. 24 has a hypersingular integral kernel and the integral does not exist in either conventional or Cauchy-principal value sense, it should be obtained in the Hadamard finite part sense [Hwang (1997), Chen J.T., Chen C.T., Chen K.H. and Chen I.L. (2000), Yan, Hung and Zheng (2003), Qian, Han and Atluri (2004)]. Discretized operator matrix \( D_0 \) for the hypersingular integral operator \( N_0 \) is

\[
D_0 = B_0^{-1}(A_0^2 - \frac{1}{4} I) \tag{25}
\]

where

\[
B_{0i,m} = \sum_{m=f(j,l)} \int \int_{A_{Sj}} N_l G_{0ij}(p,q) dS_q \tag{26}
\]

\[
A_{0i,m} = \sum_{m=f(j,l)} \int \int_{A_{Sj}} N_l \frac{\partial G_{0ij}}{\partial n} dS_q \tag{27}
\]

While for the discretized matrix \( D_{0c} \) about function \( N_{0c} \), it can be found by take the shape functions \( N_l \) in Eq. (25) equals to 1. For structures with smooth surfaces, it is easy to obtain that

\[
A_{0c} = [c(p) - 1] I = -\frac{1}{2} I \tag{28}
\]

Therefore

\[
D_{0c} = B_{0c}^{-1}(A_{0c}^2 - \frac{1}{4} I) = 0 \tag{29}
\]

Eq. (29) turns out that after regularization the term about \( N_{0c} \) is a zero matrix.

After regularization, the weakly singular integrals are solved using the scheme proposed by Lachat and Watson (1976).


3 Expression of the hypersingular operator \( N_d \)

As mentioned in the introduction, in the past decades, a lot of research has been performed on how to calculate the hypersingular integrals involved in the Burton and Miller’s formulation (1971). According to the different expressions of the hypersingular operator \( N_d \) defined by Eq. (13), these methods can be broadly divided into two kinds. For one kind, the researchers, Burton and Miller (1971), Meyer, Bell and Zinn (1978), Terai (1980), Mathews (1986), Wu, Seybert and Wan (1991), Liu and Rizzo (1992), Luke and Martin (1995), Wang, Atalla and Nicolas (1997), Liu and Chen (1999), Chen and Liu (1999), Gennaretti, Giordani and Morino (1999), Yan (2000), Yan Z.Y., Jiang, He and Yan M. (2001), Liu, Cai, Zhao, Zheng and Lam (2002), Yan, Hung and Zheng (2003) took the operator \( \partial / \partial n_p \) inside the integral directly and they had

\[
N_d \varphi = \int \int_S \frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q} \varphi(q) dS_q \tag{30}
\]

While for the other kind, the researchers, Chien, Rajiyah and Atluri (1990), Hwang (1997), Yang (1997, 1999) assumed that the normal derivative of the solid angles on the surface was zero. Based on such an assumption, they transformed the integral kernel into an integrable form and then took the differential operator into the integral. Finally, they had

\[
N_d \varphi = \int \int_S \left( \frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q} \varphi(q) - \frac{\partial^2 G_0(p,q)}{\partial n_p \partial n_q} \varphi(p) \right) dS_q \tag{31}
\]

The numerical results obtained using both of these two different expressions agreed well with the corresponding analytical solutions. Then we want to know which one is more reasonable.

From Eq. (16) and (21), we have

\[
\frac{\partial}{\partial n_p} \int \int_S \frac{\partial G_k(p,q)}{\partial n_q} \varphi(q) dS_q = \frac{\partial c(p)}{\partial n_p} \varphi(p) \\
+ \int \int_S \left[ \frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q} \varphi(q) - \frac{\partial^2 G_0(p,q)}{\partial n_p \partial n_q} \varphi(p) \right] dS_q \tag{32}
\]
\[ N_{d} \phi = \frac{\partial c(p)}{\partial n_{p}} \phi(p) + \int_{S} \left[ \frac{\partial^{2} G_{k}(p,q)}{\partial n_{p} \partial n_{q}} \phi(q) - \frac{\partial^{2} G_{0}(p,q)}{\partial n_{p} \partial n_{q}} \phi(p) \right] dS_{q} \] (33)

Even though the results obtained using Eq. (30) and (6) are the same, we must point out that both Eq. (30) and (6) are not very strict expressions for operator \( N_{d} \). The MCHIE method developed here and Eq. (33) may help the researchers to get a better understanding on the well-known hypersingular integral operator \( N_{d} \).

4 Numerical examples

Plane acoustic wave \( \varphi_{I} = \varphi_{0}e^{-ikz} \) scattering from a rigid sphere of radius \( a \) is calculated to show the correctness and feasibility of the modified Burton and Miller’s formulation with the improved regularization scheme. Eight-noded isoparametric element is applied in the computation. Some commercial FEM-codes are available to generate meshes as shown in Fig. 2. In this study, the 832-element model is generated using ANSYS with the element type SHELL93. This example is computed using the in-house developed code, SSFI (Sound Structure Fluid and their Interaction).

For plane acoustic wave scattering from a rigid sphere, the analytical solution [Junger (1952)] of the scattered acoustic pressure \( \varphi_{s}(r,\theta) \) at a distance \( r \) from the center of the sphere and at an angle \( \theta \) from the direction of the incoming wave is given by

\[ \frac{\varphi_{s}(r,\theta)}{\varphi_{0}} = \sum_{m=0}^{\infty} \left[ (-i)^{m}(2m+1)j_{m}^{'}(ka)h_{m}^{'}(ka) \right] h_{m}(kr)P_{m}(\cos \theta) \] (34)

where \( j_{m} \) is spherical Bessel function of the first kind and \( h_{m} \) is spherical Hankel function of the second kind. \( P_{m} \) denotes Legendre polynomial of order \( m \).

The surface Helmholtz integral equation (HIE) for the scattering of a plane acoustic wave from a rigid sphere is given by

\[ \left[ -\frac{1}{2}I + M_{k} \right] \varphi = \varphi_{I} \] (35)

The modified composite Helmholtz integral equation (MCHIE) for the scattering of a plane acoustic wave from a rigid sphere is given by

\[ \left[ -\frac{1}{2}I + M_{k} + \alpha(N_{k} - N_{0}) + N_{0} \right] \varphi = \varphi_{I} + \alpha \frac{\partial \varphi_{I}}{\partial n} \] (36)

where the acoustic pressure \( \varphi = \varphi_{s} + \varphi_{I} \).

Dimensionless scattered surface acoustic pressures at points \( \theta = 0 \) and \( \theta = \pi \) are presented as function of the reduced wave number \( ka \) in Fig. 3 and 4. Numerical results obtained using HIE and MCHIE are compared with the corresponding analytical solutions. It is clear that at certain characteristic frequencies the HIE can not provide unique solutions. More numerical examples to show the nonuniqueness problem can be found in the papers by Chen J.T., Chen C.T., Chen K.H. and Chen I.L. (2000), Yan, Hung and Zheng (2003), Qian, Han and Atluri (2004), Qian, Han, Ufimtsev and Atluri (2004). To show the directivity of the scattered acoustic field, dimensionless scattered acoustic pressures on the surface of the sphere are displayed in Fig. 5 to 8 as reduced wave number \( ka = 10, 4\pi, 5\pi \) and \( 6\pi \). Numerical results obtained
Investigation on the Normal Derivative Equation

Figure 3: Dimensionless scattered acoustic pressure at $\theta = 0$ as a function of $ka$

Figure 4: Dimensionless scattered acoustic pressure at $\theta = \pi$ as a function of $ka$

Figure 5: The angular dependence of scattered acoustic pressures as $ka = 4\pi$

Figure 6: The angular dependence of scattered acoustic pressures as $ka = 10$

using MCHIE are compared with the corresponding analytical solutions. Since $ka = 10$ is not a characteristic frequency, the numerical results obtained using HIE also displayed in Fig. 5 for comparison with that obtained using MCHIE and analytical solutions. Clearly, all the numerical results obtained using MCHIE agree well with the corresponding analytical solutions.

5 Conclusion
In this paper, taking the normal derivative of the solid angles on the surface into consideration, a modified Burton and Miller’s formulation (MCHIE) is derived. From which, a more reasonable expression of the hypersingu-
The angular dependence of scattered acoustic pressures as $ka = 5\pi$

Figure 7: The angular dependence of scattered acoustic pressures as $ka = 5\pi$

The angular dependence of scattered acoustic pressures as $ka = 6\pi$

Figure 8: The angular dependence of scattered acoustic pressures as $ka = 6\pi$


References


scattering from a rigid sphere is computed to validate the MCHIE with the improved regularization scheme. The numerical results obtained using MCHIE agree well with the corresponding analytical solutions.

Particular operator $N_d$ is derived. This expression will help researchers to avoid the confusion from different expressions for the same integral operator. An improved regularization formulation is successfully employed to overcome the hypersingular integrals. Plane acoustic wave


Yang, S. A. (1997): Acoustic scattering by a hard
