Thermomechanical Analysis of Functionally Graded Composites under Laser Heating by the MLPG Method

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**Abstract:** The Meshless Local Petrov-Galerkin (MLPG) method is a novel numerical approach similar to finite element methods, but it allows the construction of the shape function and domain discretization without defining elements. In this study, the MLPG analysis for transient thermomechanical response of a functionally graded composite heated by Gaussian laser beams is presented. The composite is modeled as a 2-D strip which consists of metal and ceramic phases with the volume fraction varying over the thickness. Two sets of the micromechanical models are employed for evaluating the effective material properties, respectively. Numerical results are presented for the thermomechanical responses in both the transient and steady states. A parametric study with respect to the spatial distribution and volume fraction of material constituents, the rising rate of the laser power, and the radius of the laser beam is conducted.

**keyword:** Meshless Particle Method, Functionally Graded Materials, MLPG, Thermomechanics, Laser Heating.

1 Introduction

In many engineering applications, ceramics are widely used as thermal barrier coating (TBC) in high temperature transient environments. While a ceramic coating provides corrosion, wear and erosion resistance, possesses higher compressive strength, and can protect the structural components from the severe thermal environment, joining the ceramic to a different material (e.g. metal substrate) can inevitably result in a large stress across the interface, thus often causing the delamination mode of failure in the coated structures. In this regard, one way to overcome this adverse effect is to use functionally graded materials (FGMs).

FGMs are advanced composite materials which are composed of two or more constituents. Typically, these materials are made from a mixture of ceramic(s) and metal(s). FGMs are microscopically heterogeneous, but the volume fractions of material constituents can be engineered to a continuously spatial variation, and in turn possess smoothly varying material properties. This allows FGMs to be optimized by grading the volume fractions of the material constituents for the desirable properties, and thus they can offer various advantages such as reduction of thermal stresses, minimization of stress concentration or intensity factors, and attenuation of stress waves. Therefore, FGMs have attracted considerable attention in the field of structural ceramic applications, which include gas turbines, heat-engine components, packaging encapsulants, thermoelectric generators, and human implants, just to name a few.

In the transient thermomechanical analysis of linearly elastic FGMs, numerous approaches have been proposed, either by the analytical form or by the numerical techniques. Noda (1999) solved the governing equations for thermal stresses in FGMs using a perturbation method, and delineated the crack propagation paths due to thermal shock. He also provided a literature review about the thermoelastic response of FGMs. Ueda (2001) utilized the micromechanical model to study the transient behavior of an FG divertor plate. By assuming an exponentially spatial variation of the material properties, Ootao and Tanigawa (2004, 2005) analyzed transient thermal transport and deformation for 2-D and 3-D FG plates subjected to nonuniformly convective heat supply. Jin and Paulino (2001) solved for transient thermal stresses in an FG strip by using a multi-layered material model. Vel and Batra (2003) derived a 3-D exact solution of transient thermal stresses in an FG plate under time-dependent thermal loads on its top and/or bottom surfaces. Awaji and Sivakumar (2001) employed both the variable transformation and multi-layered techniques to study the tran-

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Although analytical approaches provide closed-form solutions, they are limited to simple geometries, certain types of gradation of material properties (e.g., exponential or power law distribution), specific types of boundary conditions (e.g., simply support) and special loadings (e.g., sinusoidal loading). The above constraints can be relaxed when numerical approaches are employed, for example, finite element methods (FEMs). However, a nonhomogeneous FGM needs to be treated as numerous homogeneous elements, and thereby requiring intensive labor to generate the mesh and assign the material properties to elements. To deal with the complexity and nonhomogeneity of the properties in FGMs, a meshless method may be more suited and provides a promising cost-effective CAE tool for the design of FGM structures.

Since the last decade, the meshless particle methods have emerged as an effective numerical approach for solving initial-boundary-value problems. The feature of these methods is that only a set of scattered nodes that need not be connected to form closed polygons is required for modeling the physical domain. In contrast to the FEM, the meshless methods can save the pre-processing work of mesh generation, as no element is required in the entire model. Besides, the computed stresses and strains are smooth so that there is no need for any post-process smoothing technique. More importantly, the spatial variation of material properties in FGMs can easily be described at the level of the integration points. In the past years, a variety of meshless methods have been proposed, such as the Diffuse Element Method (DEM) (Nayroles, Touzot and Villon, 1992), the Element-Free Galerkin (EFG) method (Belytschko, Lu and Gu, 1994), the Hp-Clouds (Duarte and Oden, 1996), the Reproducing Kernel Particle Method (RKPM) (Liu, Jun and Zhang, 1995), the Partition of Unity Finite Element Method (PUFEM) (Melenk and Babuska, 1996), the Meshless Local Petrov-Galerkin (MLPG) method (Atluri and Zhu, 1998), and the Corrective Smooth Particle Method (CSPM) (Chen, Beraun and Carney, 1999). The major difference among these methods lies in the interpolation techniques. The interested readers are referred to the work by Belytschko, Krongauz, Organ and Fleming (1996), Atluri and Shen (2002), and Liu (2002) for the similarities and differences among those meshless methods.

One unique advantage of the MLPG method is that no background mesh is used to evaluate various integrals appearing in the local weak formulation of problem. Therefore, the MLPG method is a “truly meshfree” approach in terms of both interpolation of variables and integration of energy. The latest development of the MLPG method can be found in Atluri (2005). It has been demonstrated to be quite successful in solving different branches of initial-boundary-value problems. In the application of the MLPG method to FGMs, Qian and Ching (2004) studied dynamic deformation of an FG beam and found that the maximum first frequency occurs in the FG beam rather than in the homogeneous beam. Qian and Batra (2004, 2005) combined the MLPG method with a higher order plate theory to analyze the transient heat conduction and thermal stresses in a thick FG plate. Sladek J, Sladek V and Zhang (2003) and Sladek J, Sladek V, Krivacek and Zhang (2005) utilized local boundary integral equations to study elastodynamics and transient heat conduction in isotropic and anisotropic FG solids, respectively. Ching and Yen (2005) performed a steady-state thermomechanical analysis for 2-D FG solids in which the variation of material properties are either described by analytical functions or computed by micromechanical models.

Laser irradiation has been applied to examine the thermomechanical behavior of FGMs, specifically for the fracture process in a thermal barrier coating system (Takahasi, Ishikawa, Okugawa and Hashida, 1992). Elperin and Rudin (2002) developed an analytical procedure for solving transient 2-D temperature and thermal stress distribution in an FGM coating composed of tungsten carbide (WC) and steel, where the effective material properties were simply approximated by the “rule of mixture”. In this paper, we employ the MLPG method to investigate the transient thermomechanical response of a two-phase metal/ceramic FG composite subjected to high-density laser heating. The spatially various effective thermoelastic properties are evaluated by the two different homogenization schemes. The paper is organized as follows: Section 2 gives governing equations for both thermoelastic and heat conduction analyses. In Section 3, the moving least squares (MLS) approximation, the weak formu-
Thermomechanical Analysis

2 Governing Equations

For a 2-D isotropic solid occupying the domain $\Omega$ bounded by the boundary $\Gamma$ and unstressed at a reference temperature, we use rectangular Cartesian coordinates $x = \{x_1, x_2\}^T$ to describe its transient thermomechanical behaviors. The governing equations of the mechanical equilibrium in elastostatics with neglecting inertial and body forces and the transient thermal equilibrium in the absence of internal heat sources are given by

$$\sigma_{ij,j} = 0 \quad \text{in} \quad \Omega \times [0,t]$$

(1)

$$-q_{j,j} = \rho \theta \quad \text{in} \quad \Omega \times [0,t]$$

(2)

where $\sigma_{ij}$, $q_j$, $c$ and $\rho$ are the Cauchy stress tensor, the heat flux vector, the specific heat and the mass density, respectively; $\theta$ is the change in temperature with respect to the stress-free reference state. A comma followed by index $j$ denotes partial differentiation with respect to coordinate $x_j$, a superimposed dot indicates partial derivative with respect to time $t$, and a repeated index implies summation over the range of the index. Equations (1) and (2) are supplemented with the following boundary conditions:

$$u_i = \bar{u}_i \quad \text{on} \quad \Gamma_u \times [0,t]$$

(3a)

$$\sigma_{ij} n_j = \bar{\sigma}_i \quad \text{on} \quad \Gamma_t \times [0,t]$$

(3b)

and

$$\theta = \bar{\theta} \quad \text{on} \quad \Gamma_\theta \times [0,t]$$

(4a)

$$q_j n_j = \bar{q}_j \quad \text{on} \quad \Gamma_q \times [0,t]$$

(4b)

$$q_j n_j = h(\theta - \theta_c) \quad \text{on} \quad \Gamma_h \times [0,t]$$

(4c)

where $\bar{u}_i$ are the prescribed displacements on $\Gamma_u$ and $\bar{\alpha}_i$ the given tractions on $\Gamma_t$, with $\Gamma_u$ and $\Gamma_t$ being the complementary parts of the boundary $\Gamma$ (i.e., $\Gamma_u \cap \Gamma_t = \phi, \Gamma_u \cup \Gamma_t = \Gamma$). The body is also subject to the thermal conditions where the prescribed temperature $\bar{\theta}$ is specified on $\Gamma_\theta$, the given heat flux $\bar{q}_j$ is imposed on $\Gamma_q$, and the convection heat loss to an ambient temperature $\theta_c$ occurs on $\Gamma_h$. Likewise, $\Gamma_u$, $\Gamma_q$ and $\Gamma_h$ constitute another set of complementary parts of the boundary. $h$ is the coefficient of the convection, and $n_j$ are the components of the unit outward normal to $\Gamma$. Since $\theta$ equals the temperature change, the initial condition is set to be $\theta(x,0) = 0$.

The constitutive equations for a linearly isotropic thermoelastic material are as follows:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} - \beta \theta \delta_{ij}$$

(5)

$$q_j = -\kappa \theta_j$$

(6)

where $\lambda$ and $\mu$ are Lame’ constants, $\beta$ is the stress-temperature modulus, $\kappa$ is the thermal conductivity, and $\varepsilon_{ij}$ is the infinitesimal strain tensor which is related to the displacement field $u_i$ by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

(7)

In this work, the interchange between thermal and mechanical energy is neglected since the laser heating considered is not ultrafast (Chen, Tham and Beraun, 2004). Thus, the uncoupled, quasi-static thermomechanical equations are solved in the sequence: the transient temperature field is first determined by solving Eqs. (2) and (6) and the relevant boundary and initial conditions, and then the displacements are computed from Eqs. (1), (5) and (7) with the predetermined temperature field and the pertinent boundary conditions.

3 The MLPG Formulation

3.1 Brief Description of the MLS Approximation

In the MLPG method, the moving least squares (MLS) approximation is adopted for forming the basis functions $\phi_i(x)$ for an unknown trial function; see Lancaster and Salkauskas (1981) for details. For completeness, we briefly describe below the MLS approximation. Let $f^h(x,t)$ be an approximation of a scalar function $f(x,t)$

$$f^h(x,t) = p^T(x)a(x,t) = \sum_{j=1}^{m} p_j(x) a_j(x,t)$$

(8)
where \( \mathbf{p}^T(x,y) = [p_1(x), p_2(x), \ldots, p_m(x)] \) is a vector of the complete monomial basis of order \( m \). Examples of \( \mathbf{p}^T(x) \) in a 2-D problem are:

\[
\mathbf{p}^T(x) = \{1, x_1, x_2\} \quad \text{for linear basis, } m = 3 \tag{9a}
\]

\[
\mathbf{p}^T(x) = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\} \quad \text{for quadratic basis, } m = 6 \tag{9b}
\]

The \( m \) unknown coefficients \( a_j(x,t) \) are determined by minimizing a weighted discrete \( L_2 \) norm defined by

\[
J = \sum_{i=1}^{n} W(x - x_i) |\mathbf{p}^T(x_i)\mathbf{a}(x,t) - \hat{f}_i(t)|^2
\]

where \( n \) is the number of points in the neighborhood of point \( x \) for which the weight functions \( W(x - x_i) > 0 \), and \( \hat{f}_i(t) \) refers to the nodal variable at time \( t \) of the function \( f \) at point \( x_i \). We choose the following Gaussian distribution to be the weight function:

\[
W(x - x_i) = \begin{cases} 
\exp[-(d_i/c_i)^2] & \text{if } d_i \leq r_i \\
0 & \text{if } d_i > r_i
\end{cases}
\]

where \( d_i = |x - x_i| \) is the distance between points \( x \) and \( x_i \), \( c_i \) is the distance from node \( i \) to its third nearest neighboring node, and \( r_i \) is the radius of the circle outside of which \( W(x - x_i) \) vanishes.

Finding the extremum of \( J \) in Eq. (10) with respect to \( \mathbf{a}(x,t) \) leads to the following system of linear equations for the determination of \( \mathbf{a}(x,t) \):

\[
\mathbf{A}(x)\mathbf{a}(x,t) = \mathbf{B}(x) \hat{f}
\]

where

\[
\mathbf{A}(x) = \sum_{i=1}^{n} W(x - x_i)\mathbf{p}(x_i)\mathbf{p}^T(x_i) \tag{13a}
\]

\[
\mathbf{B}(x) = [W(x - x_1)\mathbf{p}(x_1), W(x - x_2)\mathbf{p}(x_2), \ldots, W(x - x_n)\mathbf{p}(x_n)] \tag{13b}
\]

Solving \( \mathbf{a}(x,t) \) from Eq. (12) and substituting it into Eq. (8), we have the following relation for the nodal interpolation

\[
\hat{f}_i(x,t) = \sum_{i=1}^{n} \phi_i(x) \hat{f}_i(t)
\]

with

\[
\phi_i(x) = \sum_{j=1}^{m} p_j(x)[\mathbf{A}^{-1}(x)\mathbf{B}](x)|_{ji}
\]

\( \phi_i(x) \) is usually called the basis function of the MLS approximation corresponding to node \( i \). Note that \( \phi_i(x_j) \) need not equal the Kronecker delta \( \delta_{ij} \), and thus \( \hat{f}_i(t) \neq f^R(x_j,t) \). For the matrix \( \mathbf{A} \) to be invertible, the number of \( n \) points must at least equal \( m \) (e.g. \( n \geq m \)). For \( m = 3 \) or 6, Chati and Mukherjee (2004) suggest that \( 15 \leq n \leq 30 \) give acceptable results for 2-D elastostatic problems. In this study, we choose \( m = 6 \) and \( k = 1 \) in Eq. (11) and take

\[
r_i = 4c_i
\]

3.2 Weak Formulation and Discretization

In this section, a weak or variational formulation corresponding to the governing equations (1) and (2) and the boundary conditions (3) and (4) is presented. The system equations are obtained by discretizing the weak formulation using the moving least squares method. We first give the weak formulation and its discrete form for the thermoelastic analysis, and the equations for the transient heat conduction analysis then can be obtained in a similar manner.

3.2.1 Thermoelastic analysis

Let \( \xi(x) = \{\xi_1, \xi_2\}^T \) be a set of test functions where \( \xi_1 \) and \( \xi_2 \) are two linearly independent functions defined in \( \Omega \). Taking the inner product of Eq. (1) with \( \xi \) and of Eq. (3a) with \( \chi \xi \), integrating the resulting equations over \( \Omega \) and \( \Gamma_u \), respectively, and adding them, utilizing the integration by parts as well as the divergence theorem, and imposing the natural boundary condition (3b) on \( \Gamma_u \), we obtain

\[
\int_{\Omega} \xi^T \sigma d\Omega - \int_{\Gamma_u} \xi^T N\sigma d\Gamma - \int_{\Gamma_u} \xi^T T d\Gamma + \chi \int_{\Gamma_u} \xi^T (u - \mathbf{p}) d\Gamma = 0
\]

where \( \chi >> 1 \) is a penalty parameter. The penalty method is chosen here for imposing essential boundary condition (3a) due to the lack of the Kronecker delta property of the basis function \( \phi_i(x) \). It has been shown that the penalty method performs with higher efficiency
than the method of the Lagrange multipliers (e.g., see Belytschko, Lu and Gu, 1994) and the orthogonal transformation technique (e.g., see Atluri, Kim and Cho, 1999). However, the selection of the penalty parameter still remains a challenge as the parameter cannot be taken ‘very large’ in order to avoid ill-conditioned of the system matrix. A suitable range for the value of the penalty parameter suggested by Zhu and Atluri (1998) is $\chi = (10^3 \sim 10^7) \cdot E$, where $E$ is Young’s modulus of the material.

The constitutive equation for thermal stresses is written in the matrix form

$$\mathbf{\sigma} = \{\sigma_{11}, \sigma_{22}, \sigma_{12}\}^T = \mathbf{D} \varepsilon - \beta \mathbf{\theta}$$  \hspace{1cm} (18)

where $\mathbf{D}$ is the matrix of elastic constants and $\beta$ the matrix of the stress-temperature moduli; both may be functions of $\mathbf{x}$. For a two-dimensional isotropic solid, $\mathbf{D}$ becomes

$$\mathbf{D} = \frac{\mathbf{E}(\mathbf{x})}{1 - \nu} \begin{bmatrix} 1 & \nu(x) & 0 \\ \nu(x) & 1 - \nu/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$  \hspace{1cm} (19)

$$\mathbf{E} = \begin{cases} E \quad & \text{for plane stress} \\ \frac{E}{1 - \nu} \quad & \text{for plane strain} \end{cases}$$  \hspace{1cm} (20)

and $\beta$ is given by

$$\beta = \beta(\mathbf{x}) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$  \hspace{1cm} (21)

$$\beta = \frac{\alpha E}{1 - \nu} \quad \text{(plane stress)}, \quad \beta = \frac{\alpha E}{1 - 2\nu} \quad \text{(plane strain)}$$  \hspace{1cm} (22)

The strain $\mathbf{\varepsilon}$ in Eq. (17) can be obtained from the following equation by replacing displacement components $u_i$ with the test functions $\tilde{\xi}_j$

$$\mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1(x)}{\partial x_1} \\ \frac{\partial u_2(x)}{\partial x_2} \\ \frac{\partial u_2(x)}{\partial x_1} + \frac{\partial u_1(x)}{\partial x_2} \end{bmatrix}$$  \hspace{1cm} (23)

The matrix $\mathbf{N}$ in Eq. (17) is

$$\mathbf{N} = \begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix}$$  \hspace{1cm} (24)

The most distinguished feature of the MLPG method is that the weak formulation is based on a local sub-domain rather than a global problem domain. We assume that $N$ nodes are placed in $\Omega$ and $S_1, S_2, \ldots S_N$ are smooth 2-D closed regions, not necessarily disjoint and of the same shape and size. Let $\Phi_1, \Phi_2, \ldots \Phi$ and $\Psi_1, \Psi_2, \ldots \Psi_N$ be two sets of linearly independent functions defined over a region, say $S_{\alpha}$. The unknown trial function $\mathbf{u}$ and the test function $\mathbf{\xi}$ can be written respectively by

$$\mathbf{u}(\mathbf{x}) = \begin{bmatrix} u_1(\mathbf{x}) \\ u_2(\mathbf{x}) \end{bmatrix} = \sum_{j=1}^{n} \Phi_j(\mathbf{x}) \mathbf{u}_j$$  \hspace{1cm} (25)

$$\mathbf{\xi}(\mathbf{x}) = \begin{bmatrix} \xi_1(\mathbf{x}) \\ \xi_2(\mathbf{x}) \end{bmatrix} = \sum_{j=1}^{n} \Psi_j(\mathbf{x}) \mathbf{\xi}_j$$  \hspace{1cm} (26)

where $\Phi_j = \phi_j \mathbf{I}$ and $\Psi_j = \psi_j \mathbf{I}$; $\mathbf{I}$ is a $2 \times 2$ identity matrix; $\mathbf{u}_j$ and $\mathbf{\xi}_j$ are $2 \times 1$ arrays. Various options of the test function leading to different MLPG formulations have been discussed by Atluri and Shen (2002). Here, we equal the test function to the weight function of the moving least squares approximation. Substitution of Eqs. (25) and (26) into (23) for $\varepsilon$ and $\mathbf{\xi}$, respectively, results in

$$\mathbf{\varepsilon} = \sum_{j=1}^{n} \mathbf{B}_j \mathbf{u}_j \quad \mathbf{\xi} = \sum_{j=1}^{n} \mathbf{B}_j \mathbf{\xi}_j$$  \hspace{1cm} (27)

and

$$\mathbf{B}_j = \begin{bmatrix} \frac{\partial \Phi_j}{\partial x_1} & 0 \\ 0 & \frac{\partial \Phi_j}{\partial x_2} \end{bmatrix} \quad \mathbf{\tilde{B}}_j = \begin{bmatrix} \frac{\partial \Phi_j}{\partial x_1} & 0 \\ 0 & \frac{\partial \Phi_j}{\partial x_2} \end{bmatrix}$$  \hspace{1cm} (28)

Replacing the domain $\Omega$ of integration in Eq. (17) by $S_{\alpha}$, substituting for $\mathbf{u}$, $\mathbf{\bar{u}}$, $\varepsilon$, and $\mathbf{\bar{\varepsilon}}$ from Eqs. (25) and (26), and requiring that the resulting equations hold for all choices of $\mathbf{\xi}_j$, we arrive at the following linear alge-
braic equations for $u_I$: 

$$
\sum_{j=1}^{n} \int_{\Omega_a} \tilde{B}^T_J DB_J \tilde{u}_J d\Omega - \sum_{j=1}^{n} \int_{\Gamma_{an}} \Psi^T_J SDS \tilde{u}_J d\Gamma 
+ \sum_{j=1}^{n} \int_{\Omega_a} \Psi^T_J S \Phi_J \tilde{u}_J d\Omega 
+ \int_{\Gamma_{an}} \Psi^T_J \tilde{t} d\Gamma + \chi \int_{\Gamma_{an}} \Psi^T J \tilde{u} d\Gamma
$$

$(I = 1, 2, \ldots, n)$

where

$$
S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}
$$

$S_i = \begin{cases} 1 & \text{if } u_i \text{ is prescribed on } \Gamma_u \\ 0 & \text{if } u_i \text{ is not prescribed on } \Gamma_u \end{cases}$

Symbolically, the simultaneous equations (29) are written as

$$K_{\alpha} u_{\alpha} = F_{\alpha}$$

(31)

The final system of equations can be obtained by repeating Eq. (28) for each of all the $N$ nodes.

The weak formulation can also be obtained from the variational form for linear elastic analysis by using the so-called Duhamel-Neumann principle (Sokolnikoff, 1956) in which the body force $b$ is replaced by $b - \beta \nabla \theta$ and the prescribed tractions $\tilde{t}$ on $\Gamma_t$ replaced by $\tilde{t} + \beta \tilde{n}$. For example, Bobaru and Mukherjee (2002) utilized this principle and gave the weak formulation of the Element Free Galerkin method for thermoelastic analysis.

3.2.2 Transient heat conduction analysis

Let $\eta(x)$ be another test function defined over $\Omega$. Following the procedure in the above thermoelastic analysis, the weak formulation associated with the governing equation (2) and the boundary conditions (4) can be written as

$$
\int_{\Omega} \nabla^T \eta q d\Omega - \int_{\Omega} \eta \rho \theta d\Omega - \int_{\Gamma_{q}} \eta n^T q d\Gamma
- \int_{\Gamma_{\theta}} \eta \bar{q} d\Gamma - \int_{\Gamma_{h}} \eta \theta d\Gamma
+ \chi \int_{\Omega} \eta (\theta - \bar{\theta}) d\Gamma = 0
$$

(32)

Here, $q$ is the vector of heat flux and is related to the Fourier heat conduction law by

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}^T = -\kappa \nabla \theta$$

(33)

with

$$\nabla \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial \theta(x)}{\partial x_1} \\ \frac{\partial \theta(x)}{\partial x_2} \end{bmatrix}$$

(34)

(29)

The unknown trial function $\theta$ and the test function $\eta$ can also be expressed in an interpolative form as

$$\theta(x) = \sum_{j=1}^{n} \phi_J(x) \tilde{\theta}_J$$

(35a)

$$\eta(x) = \sum_{i=1}^{n} \psi_J(x) \tilde{\eta}_J$$

(35b)

(30)

Substitution of Eqs. (35a) and (35b) into (34) gives

$$\nabla \theta = \sum_{j=1}^{n} C_J \tilde{\theta}_J$$

(36a)

$$\nabla \eta = \sum_{j=1}^{n} \tilde{C}_J \tilde{\eta}_J$$

(36b)

with

$$C_J = \begin{bmatrix} \frac{\partial \phi_J}{\partial x_1} \\ \frac{\partial \phi_J}{\partial x_2} \end{bmatrix}$$

(37a)

$$\tilde{C}_J = \begin{bmatrix} \frac{\partial \psi_J}{\partial x_1} \\ \frac{\partial \psi_J}{\partial x_2} \end{bmatrix}$$

(37b)

Substituting for $\theta, \eta, \nabla \theta, \nabla \eta$ from Eqs. (35) and (36) into (32) for the region $\Omega_a$ and requiring that the resulting equations hold for all choices of $\tilde{\eta}_J$, we arrive at the following system equation:

$$\sum_{j=1}^{n} M_{IJ} \tilde{\theta}_J + L_{IJ} \tilde{\theta}_J = \sum_{j=1}^{n} G_I$$

(38)

where

$$M_{IJ} = \int_{\Omega_a} \rho \phi_I \phi_J d\Omega$$

(39a)
\[ L_I = \int_{\alpha} \mathbf{C}^T \kappa \mathbf{C} d\Omega - \int_{\Gamma_{\text{ob}}} \mathbf{\psi}_I \mathbf{n}^T \mathbf{C} d\Gamma \\
+ \int_{\Gamma_{\text{ob}}} h \mathbf{\psi}_I \mathbf{\phi}_j d\Gamma - \chi \int_{\Gamma_{\text{ob}}} \mathbf{\psi}_I \mathbf{\chi} d\Gamma \]

(39b)

\[ G_I = -\int_{\Gamma_{\text{aq}}} \mathbf{\psi}_I \mathbf{\chi} d\Gamma - \chi \int_{\Gamma_{\text{ob}}} \mathbf{\psi}_I \mathbf{\chi} d\Gamma + \int_{\Gamma_{\text{ob}}} h \mathbf{\psi}_I \mathbf{\theta} d\Gamma \]

(39c)

Repeating Eq. (38) for the nodes in the entire domain leads to the system of equations.

A numerical integration is required to evaluate the domain integral on \( S_\alpha \) and the line integral on \( \partial S_\alpha \) in Eqs. (28) and (38). The region \( S_\alpha \) and the boundary \( \Gamma_{\text{aq}}, \Gamma_{\text{aq}}, \Gamma_{\text{ob}} \) and \( \Gamma_{\text{ob}} \) on \( \partial S_\alpha \) are mapped onto a \([-1,1] \times [-1,1]\) square domain and a \([-1,1]\) straight line, respectively, and the Gauss quadrature rule is utilized to numerically evaluate these integrals. Therefore, no shadow cells are needed for the purpose of integration.

We use the generalized trapezoidal rule (Cook, Malkus and Plesha, 1989) to integrate Eq. (38). The recursive relation for temperature between the time interval \( \Delta t \) is

\[ \theta_{i+\Delta t} = \theta_i + \Delta t \left\{ (1-\beta)\theta_i + \beta \theta_{i+\Delta t} \right\} \]

(40)

where \( \theta_i \) and \( \theta_{i+\Delta t} \) denote the temperature, the time derivative of temperature, respectively, at time \( t \), and \( \beta \) is a parameter that controls the stability and the accuracy of the time integration scheme. The algorithm is unconditionally stable if

\[ \beta \geq \frac{1}{2} \]

(41)

Depending on the value of \( \beta \), different time integration schemes can be obtained. In this study, the popular choice of \( \beta = 1/2 \), which is known as the Crank-Nicolson method, is employed.

4 Estimation of Effective Moduli

Analytical functions such as the exponent and power law functions are commonly used in describing the continuously varying material properties in FGMs because these functions facilitate obtaining exact solutions for the analysis of FGM structures. However, this approach may not describe the physical variation of material properties in most FGMs. Another approximation for the effective material properties of FGMs is the rule of mixtures. Again, this method does not account for the interaction between phases, and thus it only gives very approximate values for most of the effective moduli. A more theoretically sound approach is the micromechanical models, among which the Hashin-Shtrikman bounds (Hashin and Shtrikman, 1963), the Mori-Tanaka method (Mori and Tanaka, 1973), the self-consistent method (Hill, 1965), and the mean field approach (Wakashima and Tsukamoto, 1991) are the popular ones. The micromechanical approach takes account of the interactions and uses a certain representative volume element (RVE) to solve the average local stress and strain fields of the constituents of the composite. In the present study, the effective Young’s modulus, Poisson’s ratio, thermal conductivity, and the coefficient of thermal expansion, we implement two homogenization schemes, the Mori-Tanaka method and the self-consistent method, to compute the properties. For brevity, these two methods are summarized below.

4.1 Mori-Tanaka Method

The Mori-Tanaka method assumes that the matrix phase, denoted by the subscript 1, is reinforced by spherical particulate phases, denoted by the subscript 2. In this notation, \( K_1, \mu_1, \kappa_1, \alpha_1 \) represent the bulk modulus, shear modulus, thermal conductivity and coefficient of thermal expansion, respectively, \( V_1 \) the volume fraction of the matrix phase, and \( K_2, \mu_2, \kappa_2, \alpha_2 \) and \( V_2 \) the corresponding material properties and the volume fraction of the particulate phase.

For a two-phase composite, the effective bulk modulus \( K \) and shear modulus \( \mu \) derived by Mori-Tanaka (1973) are given as

\[ \frac{K - K_1}{K_2 - K_1} = \frac{V_2}{1 + (1 - V_2)(K_2 - K_1)/(K_1 + 4\mu_1/3)} \]

(42)

\[ \frac{\mu - \mu_1}{\mu_2 - \mu_1} = \frac{V_2}{1 + (1 - V_2)(\mu_2 - \mu_1)/(\mu_1 + f_1)} \]

(43)

where \( f_1 = \mu_1(9K_1 + 8\mu_1)/(K_1 + 2\mu_1) \), and \( V_1 + V_2 = 1 \). The Young’s modulus and Poisson’s ratio are related to the bulk and shear moduli by \( E = 9K\mu/(3K + \mu) \) and \( \nu = (3K - 2\mu)/(2(3K + \mu)) \), respectively. The effective thermal conductivity \( \kappa \) derived by Hatta...
and Taya (1985) is
\[
\frac{\kappa - \kappa_1}{\kappa_2 - \kappa_1} = \frac{V_2}{1 + (1 - V_2)(\kappa_2 - \kappa_1)/3\kappa_1}
\]  
(44)

and the coefficient of thermal expansion \(\alpha\) derived by Rosen and Hashin (1970) is
\[
\frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1} = \frac{1/K - 1/K_1}{1/K_2 - 1/K_1}
\]  
(45)

### 4.2 Self-consistent method

The self-consistent method assumes that each reinforcement inclusion is embedded in a continuum material whose effective properties are those of the composite. This method does not distinguish between matrix and reinforcement phases, and the same overall moduli are predicted in another composite in which the roles of the phases are interchanged. This makes it particularly suitable for determining the effective moduli in those regions that have an inter-connected skeletal microstructure.

For a two-phase composite, the effective bulk modulus \(K\) and shear modulus \(\mu\) are obtained by solving the following coupled equations (Hill, 1965):
\[
\frac{1}{K + 4\mu/3} = \frac{V_1}{K_1 + 4\mu/3} + \frac{V_2}{K_2 + 4\mu/3} 
\]  
(46)

\[
\frac{V_1 K_1}{K_1 + 4\mu/3} + \frac{V_2 K_2}{K_2 + 4\mu/3} + 5 \left( \frac{V_1 \mu_2}{\mu - \mu_2} + \frac{V_2 \mu_1}{\mu - \mu_1} \right) + 2 = 0
\]  
(47)

The self-consistent model of the thermal conductivity coefficient \(\kappa\) derived by Hashin (1968) is in the implicit form as
\[
\frac{V_1 (\kappa_1 - \kappa)}{\kappa_1 + 2\kappa} + \frac{V_2 (\kappa_2 - \kappa)}{\kappa_2 + 2\kappa} = 0
\]  
(48)

Equation (45) is used to evaluate the coefficient of thermal expansion \(\alpha\) with \(K_1, K_2, \alpha_1, \alpha_2\) and the effective bulk modulus \(K\) determined form Eqs. (46) and (47).

Figure 1 shows the effective properties as functions of the volume fraction of \(V_2\) computed with the two homogenization schemes. It can be seen that the Mori-Tanaka method results in larger values for the effective Poisson’s ratio, coefficient of thermal expansion, and thermal conductivity but smaller values for the effective Young’s modulus.

### 5 Results and Discussion

A computer code based on the aforesaid MLPG formulation was developed and used to analyze transient thermoelastic response of 2-D FG solids composing of Al metal and SiC ceramic phases. Figure 2 depicts a schematic sketch of the problem studied. The FG strip with \(L = 50 \text{ mm}\) and \(H = 10 \text{ mm}\) (i.e. length-to-thickness ratio: \(L/H = 5\)) is irradiated by a Gaussian laser beam.
Table 1: Material properties for Al and SiC

<table>
<thead>
<tr>
<th>Property</th>
<th>Al</th>
<th>SiC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus $E$ (GPa)</td>
<td>70</td>
<td>427</td>
</tr>
<tr>
<td>Poisson’s ratio $\nu$</td>
<td>0.3</td>
<td>0.17</td>
</tr>
<tr>
<td>Coefficient of thermal expansion $\alpha$ ($/K$)</td>
<td>$23.4 \times 10^{-6}$</td>
<td>$4.3 \times 10^{-6}$</td>
</tr>
<tr>
<td>Thermal conductivity $\kappa$ (W/mK)</td>
<td>233</td>
<td>65</td>
</tr>
<tr>
<td>Specific Heat $c$ (J/KgK)</td>
<td>896</td>
<td>670</td>
</tr>
<tr>
<td>Density $\rho$ (Kg/m$^3$)</td>
<td>2707</td>
<td>3100</td>
</tr>
</tbody>
</table>

Figure 2: An FG composite subjected to the Gaussian laser heat flux on the top surface

The intensity distribution $q_p$ of the absorbed laser energy is expressed as

$$q_p = I_0 f(t) \frac{1}{2\pi a^2} e^{-\frac{(x_1-L/2)^2}{2a^2}}$$

where $I_0$ is the laser beam power, $a$ is the beam radius, and the temporal function is assumed to be $f(t) = (1 - e^{-\gamma t})$ with $\gamma$ being a time rise constant. The bottom surface is thermally insulated (i.e. $q = 0$), and the two edges of the strip are simply-supported and held at a reference temperature at all times

$$u_2(0, x_2, t) = u_2(L, x_2, t) = 0;$$
$$\theta(0, x_2, t) = \theta(L, x_2, t) = 0$$

Material properties of Al and SiC are listed in Table 1. The volume fraction of the ceramic phase varies over the thickness by a power law function

$$V_c = V_c^- + (V_c^+ - V_c^-)(x_2/H)^n$$

where $V_c^+$ and $V_c^-$ are the volume fractions of SiC on the top and the bottom surfaces, respectively; $n$ is a power law index that dictates the volume fraction profile through the thickness. Effective material properties of the 2-D FG strips are evaluated by both the Mori-Tanaka method and the self-consistent method with the ceramic SiC taken as phase 2. The through-the-thickness effective properties are plotted in Fig. 3 for different values of $n$, computed with the two sets of micromechanical models respectively.

Due to symmetry of the problem about the vertically centroidal plane, only one half of the domain is analyzed under the plane stress state. Unless otherwise specified,
Figure 3: Through-the-thickness variations of the effective material moduli obtained by the Mori-Tanaka method and self-consistent method for different values of the power law index $n$; (a) Young’s modulus, (b) shear modulus, (c) coefficient of thermal expansion, and (d) thermal conductivity.

typical values are given for $V_c^+ = 1.0$, $V_c^- = 0$, $n = 2$, $\gamma = 10.0$ s$^{-1}$, $I_0 = 400$ W, $a = a_0 = 1$ mm, and the Mori-Tanaka method is a default homogenization scheme in the analyses. Numerical results are presented in terms of the nondimensional variables defined by

$$
\left[ \overline{\theta}, \overline{u_2}, \overline{\sigma_{11}}, \overline{\sigma_{12}} \right]
\begin{align*}
= 2\pi a_0^2 \frac{\kappa_{Al}\theta}{I_0} \left[ \frac{10\kappa_{Al}u_2}{\alpha_{Al}L^2}, \frac{100\kappa_{Al}\sigma_{11}}{E_{Al}\alpha_{Al}L}, \frac{1000\kappa_{Al}\sigma_{12}}{E_{Al}\alpha_{Al}L} \right]
\end{align*}

(52)
The convergence study for the nodal density and time step is first examined. Figure 4(a) compares the variation of the longitudinal stress $\sigma_{11}$ through the vertically centroidal surface of the FG strip for the four different nodal densities at $t = 2$ s. In these calculations we fix $(x_1, x_2) = L/2$ and use $17, 21, 25$ and $29$ nodes along the $\gamma = 0.1 s^{-1}$, $V_c^- = 0$, $V_c^+ = 1$ and $n = 2$

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Fig. 4(a) reveal that the simulated longitudinal stresses are almost identical for the four sets of nodal density, thus indicating that the convergence regarding the spatial discretization has been achieved. For the time step size, three different time steps $\Delta t = 10^{-2}$ s, $10^{-3}$ s and $10^{-4}$ s are chosen. It is found from Fig. 4(b) that the variations of the longitudinal stress are also indistinguishable. This means that a time step of $\Delta t = 10^{-2}$ s already gives essentially converged results. Therefore, we use a nodal mesh of $49 \times 25$ nodes and the time step size of $10^{-3}$ s in the following calculations.

Figures 5(a-d) show the time histories of temperature $\theta$, transverse displacement $u_2$, longitudinal stress $\sigma_{11}$, and transverse shear stress $\sigma_{12}$ at different points in the FG strip for $\gamma = 10.0$ s$^{-1}$, $1.0$ s$^{-1}$, and $0.1$ s$^{-1}$. It appears that for $\gamma = 10.0$ s$^{-1}$, all of the physical quantities reach the steady state rapidly while the change is more gradual for $\gamma = 0.1$ s$^{-1}$. The steady state is reached at about $t = 20$ s for $\gamma = 10.0$ s$^{-1}$, $t = 25$ s for $\gamma = 1.0$ s$^{-1}$, and $t = 50$ s for $\gamma = 0.1$ s$^{-1}$. Regardless of the values of $\gamma$, the temperature at the centroid and the longitudinal stress at the center of top surface increase monotonically to their steady state values, while this is not always the case for the transverse displacement and the transverse shear stress. For example, the transverse displacement moves upward (i.e. in the positive $x_2$ direction) in the early time, but then deflects downward and increases in magnitude until the steady state is established.

Through-the-thickness variations of the temperature, transverse displacement and longitudinal stress at $x_1 = L/2$, and the transverse shear stress at $x_1 = 0$ are plotted in Figs. 6(a-d) for time $t = 0.5$ s, $5$ s and $50$ s. Clearly, the temperature increases monotonically with its maximum values always occurring on the top (heated) surface. In general, the Mori-Tanaka micromechanical approach computes lower temperature than the self-consistent approach, especially in the region near the irradiated surface. This is because more heat diffuses away due to the larger effective thermal conductivity described by the Mori-Tanaka method (Fig 3(a)). In the early times (e.g. $t = 0.5$ s), the transverse displacement calculated with the Mori-Tanaka approach is positive but changes to negative as time increases. The change is mainly caused by bending due to the different thermal expansion ($\frac{\alpha A_0}{A}$) in the upper (hotter) and lower (colder) regions. From Figs. 3(c) and 6(a), the ratio of $\frac{\alpha A_0}{A}$ at the top and bottom surfaces is found to be about 3.0 at $t = 0.5$ s, 1.3 at $t = 1.0$
Figure 5: Time histories of the normalized (a) temperature change, (b) transverse displacement, (c) longitudinal stress and (d) transverse shear stress, computed by the Mori-Tanaka method for different values of $\gamma$ with $V_c^- = 0$, $V_c^+ = 1$ and $n = 2$

s, and 0.44 at $t = 50$ s. A greater expansion in the top portion would deflect the media upward, and thereby resulting in the positive transverse displacement. This occurs during the early times. An opposite conclusion can be made for the late times, as the greater thermal expansion occurring in the lower portion evidenced by the ratio of $\alpha \Delta \theta$ at the two surfaces becomes 0.44 at $t = 50$ s. The less pronounced bending effect found for the Mori-Tanaka method in Fig. 6(b) is due to the larger difference of the effective thermal expansion coefficient in the top region between the two methods. At these three time instants, the calculated longitudinal stress is in tension near the top and bottom surfaces and in compression in the middle region of the FG strip. The transverse shear stress consists of two half-sine waves of different amplitudes through the thickness.

Figures 7(a-d) show the dependences of the temperature,
transverse displacement, longitudinal stress and transverse stress on the power law index $n$, computed by the Mori-Tanaka method and the self-consistent method. Although the temperature obtained from the two micromechanical approaches are quite close in both the transient and steady states, the other physical quantities agree qualitatively but differ quantitatively. As shown in Fig. 7(a), the temperature at the centroid of the FG strip decreases as $n$ increases, and is always lower than that in a pure ceramic strip (i.e. $n = 0$). This is because the heat capacity ($\rho c$) of $\text{SiC}$ is smaller than that of $\text{Al}$. The lower temperature predicted by the Mori-Tanaka approach than that by the self-consistent approach is due to the larger effective thermal conductivity predicted by the former, as explained previously. The trend of monotonic decrease with $n$ for temperature does not apply to the transverse
Figure 7: Variations of the normalized (a) temperature change, (b) transverse displacement, (c) longitudinal stress, and (d) transverse shear stress with the power law index $n$, computed by both the Mori-Tanaka method and self-consistent method with $\gamma = 10.0s^{-1}$, $V_c^- = 0$, and $V_c^+ = 1$ at $t = 1s$ and $50s$.

Displacement. Instead, Fig. 7(b) shows the magnitude of the transverse displacement is in positive first and decreases with $n$, reaches the maximum negative value, and then increases. This behavior, again, can be explained by the different thermal expansion between the upper and lower regions. When the power index $n$ becomes large enough, the displacement sign will change back to positive as that seen for small $n$. The non-monotonic change with $n$ is also seen for the stresses in Figs. 7(c) and 7(d).

Figures 8(a) and 8(b) display the time histories of the temperature and transverse displacement at the centroid of the strip for four different values of $n$. Again, the temperature and deformation in the FGMs differ substantially from those of their homogeneous counterpart ($n = 0$).

With the significant discrepancy of the thermomechanical results obtained by the Mori-Tanaka method and the
Thermomechanical Analysis

Figure 8: Time histories of the normalized (a) temperature change and (b) transverse displacement computed by the Mori-Tanaka method for different values of the power law index $n$ with $\gamma = 10.0s^{-1}, V_c^- = 0,$ and $V_c^+ = 1$

dex $n$ in those quantities (c.f. Figs. 7(b) and 7(c)), the transverse displacement and the longitudinal stress are monotonic functions of $V_c^+$. At both $t = 1$ s and 50 s, the magnitude of the transverse displacement increases with the increase of $V_c^+$. The trend also holds for the longitudinal stress. The corresponding time evolutions for different values of $V_c^+$ are shown in Figs. 9(c) and 9(d), respectively.

In Figs. 10(a) and 10(b), we compare through-the-thickness variations of the transverse shear stress along the left edge for different $V_c^+$ at the early time ($t = 1$ s) and the steady state, respectively. At the steady state, the magnitude of the shear stress at a point increases as $V_c^+$ increases. In a transient state, however, the magnitude does not necessarily increases with the increase in $V_c^+$. When $V_c^+ = 0$, the FG strip is recovered to a pure aluminum strip. As seen in Fig. 10(b), no shear stress is induced in the pure aluminum strip at the steady state.

Figures 11(a) and 11(b) show the effects of the laser beam radius on the temperature and the longitudinal stress induced along the vertically centroid surface, respectively. It is evident that under the same laser power, a smaller laser beam results in greater temperature rise, thereby causing more severe thermal stresses. At $t = 1$ s, the temperature at the center of the irradiated surface rises up to $4107^\circ K (\theta = 1.503), 1567^\circ K (\theta = 0.574), 874^\circ K (\theta = 0.320)$, and $569^\circ K (\theta = 0.208)$ for the beam radius of $a = 0.5$ mm, 1.0 mm, 1.5 mm and 2.0 mm, respectively. Hence, for typical values of the parameters studied here (i.e. $a = 1$ mm and $I_o = 400$ W), one can heat the surface of the FG strip at a rate of $1000^\circ K/s$, which is usually required for the thermal reliability of specimens. Under this operating condition, the induced tensile stress is found to be $0.93 GPa (\sigma_{11} = 4.160)$. Elperin and Rudin (2002) also showed that the temperature of an FGM coating under the laser heating increases as the laser beam radius decreases. In their study, a similar heating rate of $1000^\circ K/s$ by a laser beam of power 1 kW and radius 1 $\sim 2$ mm is reported.

6 Conclusions

We have presented the MLPG analysis for thermoelastic response of a 2-D Al/AlC functionality graded composite subjected to high intensity laser irradiation. The effective material properties are evaluated from the local volume fraction of ceramic and metal phases by using two homogenization schemes. It is shown that a tai-
lored FGC can significantly alter the temperature and deformation fields as compared to those for their homogeneous counterparts. An increase of the power law index $n$ in the material composition distribution $V_c = V_c^- + (V_c^+ - V_c^-)(x_2/H)^n$ can result in a decrease of the temperature change, but the trend is not always true for the stresses. For a fixed power index $n$, both the displacement and stresses induced change monotonically with the increase of $V_c^+$, the ceramic volume fraction on the top surface. It is also found that the thermomechanical results obtained with the effective properties of FGCs computed by the Mori-Tanaka method and the self-consistent method are significantly different. Therefore, it is important to use a high fidelity model of the effective properties in the investigation of FGCs. In addition, lasers of the same power that has a smaller beam size not only results in greater temperature but also causes higher stress in the FG strip studied.

Figure 9: Variations of the normalized (a) transverse displacement and (b) longitudinal stress with volume fraction $V_c^+$; the time histories of the normalized (c) transverse displacement and (d) longitudinal stress for different values of $V_c^+$; results are computed by the Mori-Tanaka method for $\gamma = 10.0 \text{s}^{-1}$, $V_c^- = 0$, and $n = 2$. 
Unlike finite element methods, the MLPG method requires only a set of nodes for both the interpolation of the trial functions and the integration of the weak forms. Moreover, this meshfree method dictates the continuous material properties of FGMs directly to a quadrature point. These prominent features make the MLPG method well-suited in the analysis of functionally graded composite structures.

References


