where
\[ C^k_n = \begin{bmatrix} 0 & 0 & 0 & \cos \beta & \sin \beta & 0 \end{bmatrix}, \quad V^k_n = \begin{bmatrix} \sin \varphi & \sin \varphi \end{bmatrix}. \]

\( C^k_n \in \mathbb{R}^{1 \times 6} \) is a compatibility matrix for frictional support \( k \) in the normal direction, \( V^k_n \in \mathbb{R}^{1 \times 2} \) is a transformation matrix and \( \dot{\mathbf{u}}^i \in \mathbb{R}^6 \) is a vector of nodal displacement rates for element \( i \).

The governing sliding and nonpenetration conditions in Eqs. 1 and 3 for a single contact \( k \) are extended to all \( c \) frictional supports of the whole structure having \( y \) degrees of freedom, thus leading to the following relations with new indexless symbols:
\[ \pi_c = N^T_n r_n - N^T_t r_t \geq 0, \quad \dot{\mathbf{\xi}} \geq 0, \quad \pi^T_c \dot{\mathbf{\xi}} = 0, \quad (4) \]
\[ \pi_n = -C_n \dot{\mathbf{u}} - V_n \dot{\mathbf{\xi}} \geq 0, \quad r_n \geq 0, \quad \pi^T_n r_n = 0. \quad (5) \]

Matrix \( C_n \in \mathbb{R}^{c \times d} \) and vector \( \dot{\mathbf{u}} \in \mathbb{R}^d \) are assembled by using appropriate location vectors. Concatenated vectors \( \pi_c \in \mathbb{R}^{2c} \) and \( \dot{\mathbf{\xi}} \in \mathbb{R}^{2c} \) collect all \( c \) corresponding contact vectors, e.g. \( \pi^T_c \equiv [\pi^1_c, \ldots, \pi^c_c] \). Vectors \( \pi_n \in \mathbb{R}^{c}, r_n \in \mathbb{R}^{c} \) and \( r_t \in \mathbb{R}^{c} \) collect their \( c \) corresponding variables, e.g. \( \pi^T_n = [\pi^1_n, \ldots, \pi^n_c] \). Block-diagonal matrices \( N_n \in \mathbb{R}^{c \times 2c}, N_t \in \mathbb{R}^{c \times 2c} \) and \( V_n \in \mathbb{R}^{c \times 2c} \) collect all \( c \) corresponding matrices, e.g. \( N_n \equiv \text{diag}(N^1_n, \ldots, N^n_n) \).

### 2.3 Statics and kinematics

Consider frame element \( i \) with a frictional contact \( k \) at its end “\( b \)” as in Fig. 2a. For our assumed small displacement assumption, both equilibrium and compatibility relations are linear. Equilibrium between the nodal applied forces \( \alpha f^i \), the interface forces \( (r^k_n, r^k_t) \) and the elemental stress resultants \( s^i \) can then be written as
\[ C^iT s^i = \alpha f^i - C^k_n r^k_n - C^k_t r^k_t, \quad (6) \]
where
\[ C^i = \begin{bmatrix} \cos \theta & \sin \theta & 0 & -\cos \theta & -\sin \theta & 0 \\ -\sin \theta / l^i & \cos \theta / l^i & 1 & \sin \theta / l^i & -\cos \theta / l^i & 0 \\ -\sin \theta / l^i & \cos \theta / l^i & 0 & \sin \theta / l^i & -\cos \theta / l^i & 1 \end{bmatrix}, \]
\[ C^k_t = \begin{bmatrix} 0 & 0 & 0 & -\sin \beta & \cos \beta & 0 \end{bmatrix}. \]

\( C^i \in \mathbb{R}^{3 \times 6} \) is an elemental linear compatibility matrix and \( C^k_t \in \mathbb{R}^{1 \times 6} \) a compatibility matrix pertaining to frictional support \( k \) in the tangential direction.
The compatibility condition between the nodal displacement rates \( \dot{\mathbf{u}}^i \) and the plastic strain rates \( \dot{\mathbf{p}}^i \) is given by

\[ \dot{\mathbf{p}}^i = C^i \dot{\mathbf{u}}^i. \] (7)

Likewise, compatibility between the displacement rates \( \dot{\mathbf{u}}^i \) and the tangential displacement rates \( V_t^i \dot{\xi}^k \) at the sliding contact \( k \) is

\[ C_t^k \dot{\mathbf{u}}^i = V_t^i \dot{\xi}^k, \] (8)

where \( V_t^k \in \mathbb{R}^{1 \times 2} = \begin{bmatrix} \cos \phi & -\cos \phi \end{bmatrix} \) is a transformation matrix.

As is conventional, both static and kinematic relations for the complete structure can be obtained by assembly of elemental quantities. For the whole structural system discretized into \( n \) elements, \( d \) degrees of freedom, \( m \) natural generalized stresses (or strains) and \( c \) frictional supports, the linear equilibrium and compatibility relations of the structure can then be defined as follows:

\[ C^T_s = \alpha f - C^T_n r_n - C^T_t r_t, \] (9)

\[ \dot{\mathbf{p}} = C \dot{\mathbf{u}}, \] (10)

\[ C_t \dot{\mathbf{u}} = V_t \dot{\xi}. \] (11)

Matrices \( C \in \mathbb{R}^{m \times d} \), \( C_t \in \mathbb{R}^{c \times d} \) and vector \( f \in \mathbb{R}^d \) are assembled quantities. The concatenated vectors \( \mathbf{s} \in \mathbb{R}^m \), \( \dot{\mathbf{p}} \in \mathbb{R}^m \) and block-diagonal matrix \( V_t \in \mathbb{R}^{c \times 2c} \) collect all their corresponding vectors and matrices, respectively. For instance, \( \mathbf{s}^T = [s_1^T \ldots s_n^T] \), \( \dot{\mathbf{p}}^T = [\dot{p}_1^T \ldots \dot{p}_d^T] \) and \( V_t \equiv diag(V_1^t, \ldots, V_c^t) \).

### 2.4 Constitutive law

As indicated in the foregoing, the constitutive law is based on a rigid perfectly-plastic material assumption. Typically, the yield conformity condition for, say, start hinge “a” of element \( i \) (Fig. 1), subjected to combined axial and flexural forces \( (s_1^i, s_2^i) \), is nonlinear (dashed line in Fig. 3); \( s_{1u}^i \) and \( s_{2u}^i \) are respectively the corresponding yield capacities. In our work, the computationally advantageous piece-wise linear approximation to this nonlinear yield locus, as popularized by Maier and his group (see e.g. Maier (1970); Cocchetti and Maier (2003)), has been adopted.

Without undue loss of accuracy, the nonlinear yield surface is appropriately replaced by a set of a priori determined (and safe, in our case) yield hyperplanes (e.g. solid lines 1 to 8 in Fig. 3).

The general mathematical description representing the yield condition for all hyperplanes for plastic hinge “a” of an element \( i \) is written as follows:

\[ \mathbf{w}^a = -N_a^T s_a^a + r_a^a \geq 0, \quad \dot{z}^a \geq 0, \quad \mathbf{w}^a T \dot{z}^a = 0. \] (12)
With reference to the yield locus in Fig. 3, \( w^a \in \mathbb{R}^8 \) and \( \dot{z}^a \in \mathbb{R}^8 \) collect respectively the yield functions \( w_j \) and plastic multiplier rates \( \dot{z}_j \) for all hyperplanes \( j \in \{1, \ldots, 8\} \). \( N^a \in \mathbb{R}^{2 \times 8} \) contains the outward unit normals to the yield surfaces and \( r^a \in \mathbb{R}^8 \) is a vector of yield limits (distance of each hyperplane from the origin).

The complementarity conditions in Eq. 12 imply the componentwise relationship \( w_j \geq 0, \dot{z}_j \geq 0 \) and \( w_j \dot{z}_j = 0 \) for all \( j \). Mechanically, plastic yielding (\( \dot{z}_j > 0 \)) can occur only if the stress point is actually on the yield surface (\( w_j = 0 \)) and hence \( w_j \dot{z}_j = 0 \). Moreover, when the material is still rigid (\( w_j > 0 \)) there is no plastic flow (\( \dot{z}_j = 0 \)), again satisfying the complementarity condition \( w_j \dot{z}_j = 0 \).

The plastic deformation rates \( \dot{p}^a \) are defined through an associated flow rule. Thus the plastic strain rates \( \dot{p}^a \) are functions of the plastic multiplier rates \( \dot{z}^a \) as follows:

\[
\dot{p}^a = N^a \dot{z}^a. \tag{13}
\]

In essence, this normality condition ensures that the direction of the plastic deformation rate vector \( \dot{p}^a \) is normal to the yield surface, as indicated by the arrow in Fig. 3, for hyperplane 2.

The relations given in Eq. 12 and Eq. 13 can be extended to the element level by collecting relations pertaining to the two hinges “a” and “b” of each element and then to the entire structure. For a structural model consisting of \( n \) elements, \( m \) natural generalized stresses (or strains) and \( y \) plastic yield conditions, the relations can be compactly written as

\[
w = -N^T s + r \geq 0, \quad \dot{z} \geq 0, \quad w^T \dot{z} = 0, \tag{14}
\]

\[
\dot{p} = N \dot{z}, \tag{15}
\]
where indexless symbols collect all \( i \in \{1, \ldots, n\} \) elemental vectors and matrices as concatenated vectors and block-diagonal matrices, respectively. For example, \( w^T \in \mathbb{R}^y \equiv [w_1^T \cdots w_n^T] \), \( \dot{z}^T \in \mathbb{R}^y \equiv [\dot{z}_1^T \cdots \dot{z}_n^T] \), \( r^T \in \mathbb{R}^y \equiv [r_1^T \cdots r_n^T] \) and \( N \in \mathbb{R}^{m \times y} \equiv \text{diag}(N_1, \ldots, N_n) \).

3 State problem

Our aim, in line with the philosophy of a classical limit analysis problem [Kamenjarzh (1996)], is to determine the critical load factor at which plastic collapse occurs (see e.g. Carvelli, Maier, and Taliercio (2000); Leu and Chen (2006); Chen, Liu, and Cen (2008)) without the need for an expensive time-stepping analysis. Even in the presence of unilateral friction contact, we can still attempt to formulate our state problem by first simply collecting the four basic ingredients of the structural behavior, namely contact relationships (Eqs. 4-5), statics (Eq. 9), kinematics (Eqs. 10-11) and constitutive laws (Eqs. 14-15). As for classical limit analysis, we also require that the displacement rates \( \dot{u} \), which define the onset of the collapse motion, must cause positive dissipation to be produced by the live loads \( f \). This condition can be expressed in conventional normalized form as

\[
f^T \dot{u} = 1. \tag{16}
\]

All these relations, in variables \((\alpha, s, \dot{u}, r_n, r_t, \xi)\), can be collected in compact tableau format as follows:

\[
\begin{bmatrix}
\vdots & \ldots & \ldots & f^T & \ldots & \ldots \\
\vdots & \ldots & \ldots & -C & N & \ldots \\
\vdots & \ldots & \ldots & -C_n & -V_n & \ldots \\
\vdots & \ldots & \ldots & -C_t & V_t & \ldots \\
-f & C_t^T & C_n^T & \ldots & \ldots & \ldots \\
\vdots & \ldots & \ldots & -N_t^T & -N_n^T & \ldots \\
\end{bmatrix}
\begin{bmatrix}
\alpha \\
s \\
r_n \\
r_t \\
\dot{u} \\
\xi \\
\end{bmatrix}
= \begin{bmatrix}
\vdots \\
\vdots \\
r_n \\
\xi \\
\vdots \\
\vdots \\
\end{bmatrix}
= 1,
\]

\[
\mathbf{1} = 1
\]

\[
\mathbf{1} = 1
\]

\[
\mathbf{1} = 1
\]

\[
\mathbf{1} = 1
\]

\[
\mathbf{1} = 1
\]

\[
\mathbf{1} = 1
\]

\[
\mathbf{1} = 1
\]

where \((\cdot)\) represents a null vector or zero matrix of appropriate size. It is easy to recognize that Eq. 17 is a special instance of a mathematical program known as a mixed complementarity problem (MCP) [Cottle, Pang, and Stone (1992)].

In the case of associative frictional contact \((\varphi = \phi)\), normality is ensured so that \( V_n = N_n \) and \( V_t = N_t \). The MCP given by Eq. 17 now becomes the following
skew-symmetric system:

$$\begin{bmatrix}
\cdots & \cdots & \cdots & f^T & \cdots \\
\cdots & \cdots & -C & N & \cdots \\
\cdots & \cdots & -C_n & -N_n & \cdots \\
\cdots & \cdots & -C_t & N_t & \cdots \\
-f & C^T & C^T & C^T & \cdots \\
-N^T & \cdots & \cdots & \cdots & \cdots \\
N_n^T & -N_n^T & \cdots & \cdots & \cdots \\
\end{bmatrix}
\begin{bmatrix}
\alpha \\
s \\
r_n \\
r_t \\
\dot{u} \\
\dot{\xi} \\
\end{bmatrix}
-\begin{bmatrix}
\pi_n \\
\pi_c \\
\end{bmatrix} = \begin{bmatrix}
1 \\\. \\
. \\
. \\
. \\
. \\
\end{bmatrix},$$

$$\begin{align*}
\pi_n & \geq 0, \ r_n \geq 0, \ \pi_n^T r_n = 0, \ \pi_c \geq 0, \ \dot{\xi} \geq 0, \ \pi_c^T \dot{\xi} = 0, \ w \geq 0, \ \dot{z} \geq 0, \ w^T \dot{z} = 0.
\end{align*}$$

It is interesting to note that the recovery of symmetry in the associative model provides some verification that the formulation is indeed correct. In effect, the square system represents the Karush-Kuhn-Tucker (KKT) conditions of typically some extremum principle (see e.g. Maier (1970); Liu and Atluri (2008)). As for classical limit analysis, the static and kinematic variables become uncoupled, and the MCP can be recognized as being the necessary and sufficient optimality KKT conditions of a pair of dual linear programming (LP) problems with common (unique) optimal values of $\alpha$ [Ferris and Tin-Loi (2001)]. These two LP problems are in fact the well-known expressions of the bound theorems of plasticity.

More explicitly, the LP problem involving static variables encodes the static (lower bound) theorem as follows:

\[
\begin{align*}
\text{maximize} & \quad \alpha \\
\text{subject to} & \quad -\alpha f + C^T s + C_n^T r_n + C_t^T r_t = 0, \\
& \quad -N^T s + r \geq 0, \\
& \quad N_n^T r_n - N_t^T r_t \geq 0, \\
& \quad r_n \geq 0.
\end{align*}
\]

(19)

whereas the LP problem in kinematic variables encodes the kinematic (upper bound) theorem as follows:

\[
\begin{align*}
\text{minimize} & \quad r^T \dot{z} \\
\text{subject to} & \quad f^T \dot{u} = 1, \\
& \quad -C \dot{u} + N \dot{z} = 0, \\
& \quad -C_n \dot{u} + N \dot{\xi} = 0, \\
& \quad -C_t \dot{u} - N \dot{\xi} \geq 0, \\
& \quad \dot{z} \geq 0, \ \dot{\xi} \geq 0.
\end{align*}
\]

(20)

It is important to note, however, that the collapse limit obtained from such an associative assumption represents an upper bound solution to the nonassociative
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\(\phi \neq \phi\) collapse load. In fact, as indicated by Tin-Loi, Tangaramvong, and Xia (2007), the associative analysis provides the maximum upper bound value to the limit load for the nonassociative case given by the MCP in Eq. 17, albeit with possibly a different mechanism.

For a nonassociative friction law including the classical Coulomb model, the key matrix for the MCP in Eq. 17 is not skew-symmetric. At variance with the associative model, uniqueness of the collapse load multiplier is now no longer guaranteed. Any solution of the MCP yields an upper bound to the collapse limit. An important aim is then to capture the critical collapse load \(\alpha_c\), namely the smallest upper bound value. Two computational approaches aimed at achieving this are described in Tin-Loi, Tangaramvong, and Xia (2007).

The first approach rests on an enumerative scheme that maps in some fashion the solution space of the MCP by using a robust MCP solver such as GAMS/PATH [Dirkse and Ferris (1995)]; GAMS is an acronym for general algebraic modeling system [Brooke, Kendrick, Meeraus, and Raman (1998)]. For small structures, the critical solution \(\alpha_c\) can be identified directly, or obtained through a “postprocessing” phase by extrapolating an existing solution.

The second, more robust and efficient approach and one that has been used in the present work, computes the best (minimum) upper bound by direct optimization. The idea is conceptually simple. Since the general MCP given by Eq. 17 may admit multiple upper bound solutions, the critical collapse load \(\alpha_c\) can be directly computed by minimizing \(\alpha\) subject to the same constraints as provided by the MCP. This leads to the following (nonconvex) optimization problem:

\[
\begin{align*}
\text{minimize} & \quad \alpha \\
\text{subject to} & \quad \text{Eq. 17.}
\end{align*}
\]

Eq. 21 is an instance of the challenging class of mathematical programs with equilibrium constraints (MPECs) [Luo, Pang, and Ralph (1996)]. In our case, the equilibrium constraints are complementarity constraints.

Whilst an extensive theory of first and second order optimality conditions for an MPEC has been developed in Luo, Pang, and Ralph (1996), still relatively little is known about the numerical solution of practical, large-scale MPECs likely to arise in realistic applications. The most prominent feature of an MPEC, and one that distinguishes it from a standard nonlinear program, is the presence of complementarity constraints. These constraints classify this class of mathematical programs as a nonlinear disjunctive (or piecewise) program, and therefore carry with them a “combinatorial curse”. This makes the MPEC very difficult to solve, especially if one wishes, as ideally required in the present instance, to compute a global optimal solution. A branch-and-bound technique can be adopted to perform an exhaustive
enumeration in the search for a global optimum, but, as mentioned, is obviously severely limited in the size of problem it can handle. Nearly all methods proposed to date [Luo, Pang, and Ralph (1996)] are aimed at finding stationary solutions and/or local optimum, and are categorized roughly by the way the complementarity condition is handled.

In spite of the various theoretical difficulties, we have had considerable success in solving MPECs [Tangaramvong and Tin-Loi (2008); Tangaramvong and Tin-Loi (2009b)] similar to the one given by Eq. 21. Our methodology is to simply transform the MPEC into a standard nonlinear programming (NLP) problem by using some suitable parametric (in $\mu$) functions. The reformulated MPEC is then solved as a series of NLP subproblems that aim to increasingly achieve complementarity as the positive parameter $\mu$ is either increased or decreased. The attraction of this scheme is that each subproblem is a standard NLP problem, and general purpose industry standard NLP codes, such as GAMS/CONOPT [Drud (1994)], can be employed.

We have used three basic algorithms, all involving some treatment of the complementarity constraints, to solve the MPEC given in Eq. 21. They are briefly outlined below; for simplicity of exposition, the complementarity condition is denoted by $a \geq 0$, $b \geq 0$ and $ab = 0$.

(a) **Penalization:** The complementarity term is transferred to the objective function and penalized (see e.g. Ferris and Tin-Loi (1999)). In particular, the term $\mu ab$ is added to the objective function. The algorithm then tends to satisfy complementarity by increasing the penalty parameter $\mu$ at each NLP iterate.

(b) **Smoothing:** The condition $ab = 0$ is replaced by some smoothing function $\psi_\mu(a, b) = 0$. The particular one we use is the well-known Fischer-Burmeister function [Kanzow (1996)]

$$
\psi_\mu(a, b) = \sqrt{a^2 + b^2 + 2\mu} - (a + b). \tag{22}
$$

This function $\psi_\mu$ has the property that $\psi_\mu(a, b) = 0$ if and only if $a \geq 0$, $b \geq 0$ and $ab = \mu$. The parameterization $\psi_\mu$ is a smoothing of the mapping $\psi_{\mu=0}$ implying that it is differentiable for nonzero $\mu$. The algorithm then solves a series of NLP subproblems that iteratively decrease the smoothing parameter $\mu$ in order to drive the complementarity term to zero (see e.g. Tin-Loi (1999)).

(c) **Relaxation:** The complementarity constraint is replaced by its relaxed version $ab \leq \mu$. The relaxed problem is solved for successively smaller values of $\mu$ to force the complementarity term to approach zero (see e.g. Ferris and Tin-Loi (2001)).

In view of its noticeably better robustness, we have adopted the penalty approach for the present study. This involves solving iteratively the following NLP subprob-
lem:

\begin{align*}
\text{minimize} \quad & \alpha + \mu \left( w^T \dot{z} + \pi_c^T \dot{\xi} + \pi_n^T r_n \right) \\
\text{subject to} \quad & -\alpha f + C^T s + C_n^T r_n + C_t^T r_t = 0, \\
& -C \dot{u} + V \dot{\xi} = 0, \\
& f^T \dot{u} = 1, \\
& w = -N^T s + r \geq 0, \quad \dot{z} \geq 0, \\
& \pi_c = N_n^T r_n - N_t^T r_t \geq 0, \quad \dot{\xi} \geq 0, \\
& \pi_n = -C_n \dot{u} - V_n \dot{\xi} \geq 0, \quad r_n \geq 0,
\end{align*}

(23)

for successively higher values of \( \mu \) until a preset tolerance on the complementarity condition has been met (e.g. \( w^T \dot{z} + \pi_c^T \dot{\xi} + \pi_n^T r_n \leq 10^{-6} \)). Typical starting values of \( \mu \) are within the range 0.1 to 10, and are updated after each NLP solve by \( \mu = 10 \mu \).

4 Optimal synthesis

The focus of this section, indeed of the present paper, is to carry out the optimal synthesis of a rigid perfectly-plastic frame with unilateral frictional supports. For a given design load \( \alpha_d \) and fixed structural topology, we thus aim to find a safe material distribution such that the total weight (or volume) of the structure is a minimum.

The weight of the structure is proportional to its volume and is thus given by \( \bar{l}^T A \) (see e.g. Lamberti and Pappalettere (2007)), where \( \bar{l}^T \in \mathbb{R}^n = [l_1, \ldots, l_n] \) and \( A^T \in \mathbb{R}^n = [A_1, \ldots, A^n] \) collect respectively all \( n \) corresponding member lengths and cross-sectional areas.

For practical reasons, the areas \( A \) are not necessarily independent, such as when groups of members are required to be the same. These requirements are typically imposed through the following so-called “technological” constraints [Polizzotto (1975); Ferris and Tin-Loi (1999)]:

\begin{equation}
A = Ta,
\end{equation}

(24)

where \( T \in \mathbb{R}^{n \times t} \) is a self-evident matrix relating \( A \) to the independent cross-sectional area (positive) variables \( a \in \mathbb{R}^t \). For instance, \( T \) is an identity matrix \( (n \times n) \) when all areas \( A \) are independent, implying no technological constraints (\( a = A \) and \( t = n \)); and \( T \) is a unit vector \( (n \times 1) \) when \( A^1 = A^2 = \cdots = A^n = a \), thus with \( t = 1 \).

Furthermore, minimum/maximum area requirements can be imposed through the following bounds:

\begin{equation}
a_{lo} \leq a \leq a_{up},
\end{equation}

(25)
where \( a_{lo} \in \mathbb{R}^t \) and \( a_{up} \in \mathbb{R}^t \) are lower and upper limits on \( a \), respectively.

For a known section property type (i.e. whether an I-section, rectangular section, etc.) it is possible, as shown in Section 5, to express both the normality matrix \( N \) and the yield limit vector \( r \) of the constituent members as functions \( N(a) \) and \( r(a) \) of \( a \).

As for the extended limit analysis problem given by Eq. 21 it is almost intuitive to formulate the synthesis problem as the following MPEC in \((a, s, \dot{u}, \dot{z}, r_n, r_t, \dot{\xi})\):

\[
\begin{align*}
\text{minimize} & \quad l^T Ta \\
\text{subject to} & \quad -\alpha f + C^T s + C^T_n r_n + C^T_t r_t = 0, \\
& \quad -C\dot{u} + N(a)\dot{z} = 0, \\
& \quad -C_t\dot{u} + V_t\dot{\xi} = 0, \\
& \quad f^T \dot{u} = 1, \\
& \quad w = -N(a)^T s + r(a) \geq 0, \quad \dot{z} \geq 0, \quad w^T \dot{z} = 0, \\
& \quad \pi_c = N^T_n r_n - N^T_t r_t \geq 0, \quad \dot{\xi} \geq 0, \quad \pi_c^T \dot{\xi} = 0, \\
& \quad \pi_n = -C_n\dot{u} - V_n\dot{\xi} \geq 0, \quad r_n \geq 0, \quad \pi_n^T r_n = 0, \\
& \quad a_{lo} \leq a \leq a_{up}, \quad a \geq 0.
\end{align*}
\]

Unfortunately, as will be explained later, solution of Eq. 26 may well provide an unsafe solution for the nonassociative case.

Moreover, the particular MPEC described by Eq. 26 is not only nonconvex and nonsmooth, but it is also often discontinuous, similar to the state problem [Tin-Loi, Tangaramvong, and Xia (2007)].

Let us first examine the simpler and classical associative case. As required by duality, the state problem can be solved by any one of the pair of LP problems (Eq. 19 or Eq. 20) since uniqueness of the collapse load is guaranteed. In turn, for the synthesis problem, we do not need to solve the difficult MPEC given by Eq. 26. Instead it is possible and, in fact, preferable to formulate the minimum weight problem from either the LP problem Eq. 19 or Eq. 20. For instance, using Eq. 19, the following minimum weight problem in static variables can be cast:

\[
\begin{align*}
\text{minimize} & \quad l^T Ta \\
\text{subject to} & \quad -\alpha f + C^T s + C^T_n r_n + C^T_t r_t = 0, \\
& \quad -N(a)^T s + r(a) \geq 0, \\
& \quad N^T_n r_n - N^T_t r_t \geq 0, \\
& \quad a_{lo} \leq a \leq a_{up}, \quad a \geq 0, \quad r_n \geq 0.
\end{align*}
\]

This is a standard convex NLP problem for which an optimal and safe solution is expected.
The nonassociative case is far more complex. More explicitly, both the state problem MPEC (Eq. 21) and the synthesis problem MPEC (Eq. 26) may have nonunique solutions. In the limit analysis case, we correctly “minimize” to obtain the best (minimum) upper bound collapse load. However, for the minimum weight design, we incorrectly “minimize” since the optimization process will pick, for given $\alpha_d$, the material distribution of least weight when clearly it requires the solution with greatest weight to support the same load.

It is also useful to note that, for the nonassociative case, solving either Eq. 26 or Eq. 27 will provide the same areas $\mathbf{a}$ (and hence weight), albeit with different mechanisms. This is due to the fact that, as noted for the nonassociative state problem, the solution of Eq. 18 represents the maximum upper bound to the collapse load.

The iterative algorithm we propose to perform the optimal synthesis in the case of nonassociative conditions is a simple one and is based on the foregoing remarks. In essence, (a) we first solve Eq. 27 to obtain a (typically unsafe) solution, (b) we then process the state problem Eq. 21 to estimate the actual load capacity, (c) we scale the design load $\alpha_d$ and return to Step (a) and iterate until the state problem provides a collapse load that is the same (within some tolerance) as the initial given design load $\alpha_d$. The algorithm is summarized in the following.

1. Initialize: specify design load ($\alpha_d$), maximum number of iterations ($\maxit$) and positive convergence tolerance ($\ctol$).
2. Set: $i = 1$ and $\alpha_i = \alpha_d$.
3. Compute the independent areas $\mathbf{a}$ by solving the NLP problem given by Eq. 27 for $\alpha = \alpha_i$. Then, form the designed structure.
4. Calculate the actual collapse load $\alpha_c$ of the structure by solving the MPEC given by Eq. 21.
5. Check convergence: if either $\text{res} \equiv |\alpha_c - \alpha_d| \leq \ctol$ or $i = \maxit$, record the design variables $(\mathbf{a}, \mathbf{s}, \dot{\mathbf{u}}, \dot{\mathbf{z}}, \mathbf{r}_n, \mathbf{r}_t, \dot{\mathbf{\xi}})$, and then terminate.
6. Else, update: $i = i + 1$ and $\alpha_i = \alpha_{i-1} (\alpha_d / \alpha_c)$. Go to Step 3.

5 Illustrative examples

Three examples are described in this section to illustrate application of the proposed optimal synthesis approach. The first one considers the simple two bay frame used by Tin-Loi, Tangaramvong, and Xia (2007) for extended limit analysis under pure bending plastic hinges. It admits multiple solutions and is thus suitable as an introductory illustration of the behavior of our iterative algorithm when considered as
a synthesis problem. The second example is a more practical multistory frame. In the final example, we consider the case of a realistic double-layer 3-D roof truss. In all cases, the nonassociative Coulomb friction law ($\phi = 0$) with a friction coefficient of $\tan \phi = 0.3$ was adopted for all frictional supports.

The optimal plastic synthesis algorithm has been implemented as a MATLAB code, linked to the GAMS mathematical programming environment by a MATLAB-GAMS interface [Ferris (1998)]. The NLP solves were carried out using the robust GAMS/CONOPT solver [Drud (1994)]. Our preferred MPEC penalty algorithm was adopted to process the MPECs with $\mu = 1$, updates of $\mu = 10\mu$, and termination criterion of $w^T \dot{z} + \pi^T \dot{\xi} + \pi^T r_n \leq 10^{-6}$. The initial parameters employed for our synthesis iterative scheme in all examples are $\maxit = 30$ and $\ctol = 10^{-6}$.

### 5.1 Plane frame examples

The geometric and material properties of the sections belong to standard universal beam (UB) and universal column (UC) open I-sections with grade 300 steel [Syam (1999)]; UB sections were adopted for all beams and UC sections for all columns. Before we could set up the synthesis problem we had to develop (a) realistic expressions for member flexural ($s_{2u}^i$) and axial ($s_{1u}^i$) capacities as functions of their cross-sectional areas $a$, and (b) reasonable piecewise linear approximations to the nonlinear yield surfaces.

We first plotted, as in Figs. 4 (UB) and 5 (UC), the actual capacities $s_{2u}^i$ and $s_{1u}^i$ of all 28 UB sections (ranging from 150UB14.0 to 610UB125) and of all 13 UC sections (ranging from 100UC14.8 to 310UC158) against their cross-sectional areas. Approximate and adequate relationships were obtained by simple curve fitting. In the present case, the expressions for the UB sections (in kN and m units) are

$$s_{2u}^i = 507360a^{1.5}, \quad (28)$$
$$s_{1u}^i = 265188a, \quad (29)$$

and for UC sections

$$s_{2u}^i = 184128a^{1.4}, \quad (30)$$
$$s_{1u}^i = 280000a. \quad (31)$$

These continuous expressions are also shown in Figs. 4 and 5.

In these frame examples, we also imposed area restrictions (Eq. 25) to reflect the range of available sections, namely for UB sections: $a_{lo} = 1780 \times 10^{-6}$ m$^2$ (150UB14.0), $a_{up} = 16000 \times 10^{-6}$ m$^2$ (610UB125), and for UC sections: $a_{lo} = 1890 \times 10^{-6}$ m$^2$ (100UC14.8), $a_{up} = 20100 \times 10^{-6}$ m$^2$ (310UC158).
Figure 4: Properties of 28 steel UB sections.

Figure 5: Properties of 13 steel UC sections.
As for appropriate piecewise linearizations of the combined axial and flexural force yield interaction, we adopted the commonly used hexagonal approximations [Massonnet and Save (1965)].

In Fig. 6 we compare, in nondimensional form, the piecewise linear yield hyperplanes for all UB and UC sections (solid lines) with the actual nonlinear yield conditions (dashed lines). Clearly, the hexagonal representation provides a very good approximation; the inclination $\gamma$ of sloping yield hyperplanes is given by $\tan \gamma = 1/0.85$, and a reduction of pure bending capacity occurs when the axial force reaches a fraction ($r_b = 0.15$) of the pure axial capacity.

The piecewise linear hexagonal yield condition for a generic hinge “$a$” of a UB (or UC) steel member $i$ under combined stresses is thus

$$
\begin{align*}
\mathbf{w}^a & = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = - \begin{bmatrix} 0 & 1 \\ \hat{n} & 1 \\ \hat{n} & -1 \\ 0 & -1 \\ -\hat{n} & 1 \end{bmatrix} \begin{bmatrix} s_{2u}^i \\ s_{1u}^i \end{bmatrix} + \begin{bmatrix} s_{2u}^i \\ \tau s_{2u}^i \\ s_{2u}^i \\ \tau s_{2u}^i \\ \tau s_{2u}^i \\ \tau s_{2u}^i \end{bmatrix}, \\
\dot{\mathbf{z}}^a & = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \\ \dot{z}_6 \end{bmatrix},
\end{align*}
$$

(32)

where $\hat{n} = (s_{2u}^i / s_{1u}^i) \tan \gamma$, and $\tau = 1 + r_b \tan \gamma$. Yield functions $w_1$ to $w_6$ govern yielding of hyperplanes 1 to 6, respectively.
Using Eqs. 28-32, it is now straightforward to write the normality matrix $N^a$ and the yield limits $r^a$, as required in the optimal synthesis problem given by Eq. 27, as functions of the unknown area variables $a$.

Yielding caused by pure bending can be simply recovered from Eq. 32 by retaining the first and fourth rows, which define respectively positive and negative yielding under bending.

The constitutive relations for a typical frame element $i$ can be generated by collecting the corresponding relations for the two hinges “a” and “b”. For uniform sections, as used, hinge “a” is identical to the hinge “b” so that the complementarity yield relations at the element level are as follows:

$$w_i = -N_i^T s^i + r^i \geq 0, \dot{z}^i \geq 0, w_i^T \dot{z}^i = 0,$$

where

$$w_i^T = \begin{bmatrix} w^a & w^b \end{bmatrix}, s_i^T = \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix},$$

$$\dot{z}_i^T = \begin{bmatrix} \dot{z}_a & \dot{z}_b \end{bmatrix}, \ r_i^T = \begin{bmatrix} r^a & r^b \end{bmatrix}.$$
tional supports 6 and 8. Our aim was to design this rigid plastic structure for the load $\alpha_d = 50$ kN. The same UB section of area $a_1 \text{ m}^2$ was adopted for all beams and the same UC section of area $a_2 \text{ m}^2$ for all columns.

Two cases were investigated: (a) Case a: pure bending yield condition, and (b) Case b: combined stresses yield condition.

Our iterative algorithm successfully processed the pure bending Case a in 22 iterations. The first iteration gave a total volume $W = 16580.18 \times 10^{-6} \text{ m}^3$, area $a_1 = 1780 \times 10^{-6} \text{ m}^2$ and area $a_2 = 1966.73 \times 10^{-6} \text{ m}^2$. For this structure, an extended limit analysis (using Eq. 21) showed that it would actually collapse at $\alpha_c = 48.3547$. For the second iteration, the applied load was thus increased to $\alpha_2 = \alpha_1 (\alpha_d / \alpha_c) = 51.7012$, and the process was repeated until convergence. The convergence history is shown in Fig. 8. The final optimal design was $W = 16704.51 \times 10^{-6} \text{ m}^3$, $a_1 = 1780 \times 10^{-6} \text{ m}^2$ and $a_2 = 2008.17 \times 10^{-6} \text{ m}^2$. A check of its collapse load showed that it would indeed fail at $\alpha_c = \alpha_d = 50$, through the mechanism drawn in Fig. 9a involving translation of all frictional supports and the formation of perfectly-plastic hinges formed at both ends of the central column.

In Case b, no iteration was required to produce the minimum weight design of $W = 16851.90 \times 10^{-6} \text{ m}^3$, $a_1 = 1780 \times 10^{-6} \text{ m}^2$ and $a_2 = 2057.30 \times 10^{-6} \text{ m}^2$. Solution of the state problem confirmed that $\alpha_c = \alpha_d = 50$. The corresponding mechanism is shown in Fig. 9b and involved translation at both sliding supports, combined stress hinges at both ends of the central column, and flexural hinges at the midspan of the longer beam.
Example 2: Multistory frame

Example 2 is the more realistic multistory frame shown in Fig. 10, with Coulomb frictional supports 1 and 2, and fully fixed supports 3 and 4.

The structure had to be designed to resist an external applied load of $\alpha_d = 30$ kN. For practical reasons, a UC section with area $a_1$ m$^2$ was adopted for columns 1 to 8, a UC section with area $a_2$ m$^2$ for columns 9 to 11, a UB section with area $a_3$ m$^2$ for beams 12 to 23, and a UB section with area $a_4$ m$^2$ for beams 24 to 27. All beams were assumed to form only pure bending plastic hinges whilst all columns were allowed to yield through combined stresses.

The adopted model consists of 27 members, 23 nodes and 63 degrees of freedom.

Our proposed optimal synthesis algorithm terminated successfully after only 4 iterations with the result $W = 426883.34 \times 10^{-6}$ m$^3$, $a_1 = 7258.07 \times 10^{-6}$ m$^2$, $a_2 = 3396.21 \times 10^{-6}$ m$^2$, $a_3 = 4287.60 \times 10^{-6}$ m$^2$ and $a_4 = 2524.24 \times 10^{-6}$ m$^2$.

An extended limit analysis verified that $\alpha_c = \alpha_d = 30$. The corresponding collapse mechanism is displayed in Fig. 11.

5.2 3-D truss example

For the 3-D truss examples solved, we adopted standard circular hollow (CHS, grade C250) sections [Syam and Narayan (1999)] for all truss elements. The range of sections available was: $a_{lo} = 820 \times 10^{-6}$ m$^2$ (76.1×3.6CHS6.44) to $a_{up} = 2710 \times 10^{-6}$ m$^2$ (165.1×5.4CHS21.3). The capacity $s_{1u}$ versus area $a$ relationship for these CHS sections (in kN and m units) can be simply approximated by the following linear relation:

$$s_{1u} = 250000a.$$  \hfill (34)
The complementarity yield conditions describing the single stress model for a typical truss element $i$ are as follows:

$$
w^i = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} s^i_1 \\ s^i_{1u} \end{bmatrix}, \quad \dot{z}^i = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix},
$$

(35)

$$w^i \geq 0, \quad \dot{z}^i \geq 0, \quad w^iT\dot{z}^i = 0.
$$

Yield functions $w_1$ and $w_2$ control plasticity of a member in tension and compression, respectively.

**Example 3: 16 m × 16 m double-layer 3-D roof truss**

A specific example of a 3-D truss is the double-layer roof structure shown in Fig. 12. It is 16 m × 16 m in plan size and 2.828 m high. The truss was restrained only at some bottom layer nodes, namely in the y-axis direction along the perimeter and in all directions at four corners. At these bottom layer nodes, unilateral (along
Figure 11: Example 2: collapse mechanism (○ denotes perfectly-plastic hinge).

the x-axis) Coulomb frictional supports were installed along the perimeter, namely at supports 2, 3, 5, 8, 9, 12, 14 and 15.

The roof truss had to be designed to carry an external load of $\alpha_d = 20$ kN applied at top layer nodes, namely $F(x,y,z) = (16\alpha, 4\alpha, -16\alpha)$ at each node shown by ○ in Fig. 12, and $F/2$ and $F/4$ at the nodes indicated by • and ◦, respectively.

It was assumed that all bottom layer members had the same CHS section with area $a_1 \text{ m}^2$, all top-layer members similarly had the same CHS section with area $a_2 \text{ m}^2$, and all diagonal members were identical with CHS sections of area $a_3 \text{ m}^2$.

The discrete structural model consists of 128 members, 41 nodes and 103 degrees of freedom.

The proposed optimal plastic synthesis algorithm efficiently and successfully processed the problem in only 2 iterations. The optimal design achieved was $W = 88932.18 \times 10^{-6} \text{ m}^3$, $a_1 = 2044.62 \times 10^{-6} \text{ m}^2$, $a_2 = 1092.77 \times 10^{-6} \text{ m}^2$ and $a_3 = 2026.59 \times 10^{-6} \text{ m}^2$. The corresponding collapse mechanism is shown in Fig. 13 and involves translation at 6 contacts (namely supports 2, 3, 5, 8, 9 and 12) and no translation at 2 contacts (supports 14 and 15).

6 Concluding remarks

This paper is concerned with an important and difficult class of optimum synthesis problems involving rigid perfectly-plastic structures with some or all of its supports in unilateral frictional contact. For nonassociative contact conditions, the minimum weight design problem, illustrated without undue loss of generality by
Figure 12: Example 3: 16 m × 16 m double-layer 3-D roof truss (○, ● and ⊙ denote applied forces $F$, $F/2$ and $F/4$, respectively).

bar structures, cannot be formulated as a single optimization problem. An attempt to do so, in the same manner as its recently developed [Tin-Loi, Tangaramvong, and Xia (2007)] extended limit analysis counterpart, will lead to an unsafe design when the underlying MPEC has multiple solutions which is invariably the case.

A simple, yet robust, iterative scheme to solve this problem for practical, often large-scale structures, is proposed. The idea is to use in tandem a convex NLP formulation for design of the associative case and a nonconvex but tractable MPEC formulation for limit analysis of the actual nonassociative structure. In the process the applied load is changed iteratively until the collapse load is the same as the target design load. A number of examples, three of which are given in this paper, have been solved, all successfully, to validate the approach.
Some interesting and useful extensions of the present work include: consideration of other structural types coupled with geometric nonlinear effects; the inclusion of elasticity and concomitant limits on deflections; alternative solution procedures based on enumerative techniques from global optimization; and application to structural topology optimization.

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