The Mode Relation for Open Acoustic Waveguide Terminated by PML with Varied Sound Speed

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Abstract: An acoustic waveguide with continuously varying sound speed is discussed in this paper. When the waveguide is open along the depth, the perfectly matched layer (PML) is used to terminate the infinite domain. Since the sound speed is gradually varied, the density is assumed as constant in each fluid layer. For this waveguide, it is shown that the mode relation is derived by using the differential transfer matrix method (DTMM). To solve leaky and PML modes, Newton’s iteration is applied, and Chebyshev pseudospectral method is used for obtaining initial guesses. The solutions are with high accuracy.

Keywords: acoustic waveguide, perfectly matched layer, differential transfer matrix method, Chebyshev pseudospectral method, PML modes, leaky modes.

1 Introduction

In underwater acoustics, the problem can be described by a waveguide limited above by the sea surface. The sound speed in the ocean depends on the temperature, salinity, and pressure. In most cases, it can be regarded as a function of depth [Jensen, Kuperman, Porter, and Schmidt (2011)]. However, in the bottom layer, little change in sound speed can be observed, and it is assumed to be a constant. Like the Pekeris waveguide [Stickler and Ammicht (1980); Zhu and Lu (2008); Lu and Zhu (2007)], in which the bottom is more realistically represented by an infinite fluid halfspace, allowing for energy to be transmitted across the water-bottom interface, it introduces an additional loss mechanism to the waveguide propagation. To solve this unbounded problem, some excellent absorbing methods can be applied [Bienstman and Baets (2002)]. A perfectly matched layer (PML), introduced by Berenger in 1994 [Berenger (1994)], is a widely used tool to truncate the unbounded domain. It is an additional layer around the interested domain, in which

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the solutions decay. Mathematically, it is equivalent to a complex coordinate transformation, which is applied inside the additional region. Due to the addition of PML, the eigenfunctions of the resulting system lose the property of orthogonality, and the complete mode expansion is replaced by the infinite sum of a set of discrete modes, which can be classified into propagation modes, leaky modes and PML modes. Although the leaky modes and PML modes are not members of the spectrum of this unbounded lossless waveguide, they are useful to approximate the continuum of radiation modes, and have a better sense of the practical application. And the convergency property of the modes with PML has already been proved in [Olyslager (2004)].

The eigenvalue problem is very complicated to solve, with the help of the transfer matrix method (TMM) [Vigran (2010)], it can be transformed into a transcendental equation, whose roots correspond to the eigenmodes. The equation is also named as the mode relation.

For multilayered structures, TMM is sufficient to derive the mode relation. But when the medium is continuously varying, it fails. If the sound speed in the water layer is a continuous function, there is no analytical expression of the solutions, and the mode relation can not be expressed easily. The differential transfer matrix method (DTMM) is proposed for the computation of optical waveguide with arbitrary refractive index in [Khorasani and Mehrany (2003); Zhu and Shen (2011); Zariean, Sarrafi, Mehrany, and Rashidian (2008)]. It is a generalization of TMM, and is also useful in acoustic waveguides.

In this paper, we consider a two-layered waveguide, which contains a water layer with continuously varying sound speed and an infinite bottom halfspace. A PML is introduced to truncate the infinite domain, and DTMM is also used to derive the mode relation of leaky and PML modes. Once the mode relation is obtained, Newton’s iteration can be applied, where the initial guesses are given by Chebyshev pseudospectral method [Trefethen (2000, 2007); Driscoll, Bornemann, and Trefethen (2008)].

This paper is organized in the following manner. The mathematical formulation of unbounded acoustic waveguide with varied sound speed is presented in Section 2, the numerical methods are also given here. Numerical examples are given in Section 3. Finally, we leave the conclusion in Section 4.

2 Model and Methods

2.1 Mathematical Model of Waveguide

Considering an unbounded acoustic waveguide with continuously varying sound speed, which is \( c_1(z) \) for water \((0 < z < G)\) and \( c_2 \) for the bottom \((z > G)\), the
following two-dimensional Helmholtz equation is discussed:

\[
\begin{cases}
  u_{xx} + u_{zz} + \kappa^2 u = 0, \quad z > 0 \\
  \lim_{z \to G^-} \frac{1}{\rho} u_z = \lim_{z \to G^+} \frac{1}{\rho} u_z, \quad \lim_{z \to G^-} u = \lim_{z \to G^+} u, \\
  u|_{z=0} = 0, \quad \lim_{z \to +\infty} u(x, z) = 0,
\end{cases}
\]

where \( z \) is the depth, \( x \) is the range, and

\[\kappa(z) = \frac{\omega}{c(z)} = \begin{cases} \kappa_1(z), & 0 < z < G \\ \kappa_2, & G < z \end{cases};\]

the density is piecewise constant as

\[\rho(z) = \begin{cases} \rho_1, & 0 < z < G \\ \rho_2, & G < z \end{cases},\]

where \( \kappa_1 \) and \( \kappa_2 \) are wave numbers in different fluid layers, \( \rho_1 \) and \( \rho_2 \) are corresponding densities. The waveguide structure is shown in Fig.1. Here we suppose the acoustical waveguides are range-independent, that is \( \kappa \) only depends on the depth \( z \). The general solution of (1) has the form \( u = \phi(z)e^{i\beta x} \), in which \( \phi \) and \( \lambda = \beta^2 \) satisfy the following eigenvalue problem:
\[
\begin{aligned}
\phi_{zz} + \kappa^2 \phi &= \lambda \phi, \quad z > 0 \\
\lim_{z \to G^-} \frac{1}{\rho} \phi' = \lim_{z \to G^+} \frac{1}{\rho} \phi', \quad \lim_{z \to G^-} \phi = \lim_{z \to G^+} \phi, \\
\phi|_{z=0} = 0, \quad \lim_{z \to +\infty} \phi(z) &= 0.
\end{aligned}
\]

(2)

To solve the Helmholtz equation (1) or further approximations numerically, the depth \(z\) must be truncated. Under the assumption that bottom is homogeneous for \(z > G\), the PML technique is introduced in the depth direction. It is equivalent to a complex coordinate transformation, that is \(\hat{z} = z + i \int_0^z \sigma(t) \, dt\), in which \(\sigma(t)\) is the absorbing function of the PML. If the interested interval is \(0 < z < H\) for some \(H > G\), the PML is added on the boundary \(z = H\), and is truncated at \(z = D\) where \(D > H\). For \(0 \leq z < H\), \(\sigma(z) = 0\), and \(\sigma(H) = 0, \sigma'(H) = 0\) must be satisfied to ensure the correctness of discretization. Defining an operator \(L\) by

\[
L \phi = \frac{\rho(z)}{s(z)} \frac{d}{dz} \left[ \frac{1}{\rho(z) \cdot s(z)} \frac{d\phi}{dz} \right] + \kappa^2(z) \cdot \phi,
\]

the problem (2) is therefore approximated by the following form:

\[
\begin{aligned}
L \phi = \lambda \phi, \quad 0 < z < D \\
\lim_{z \to G^-} \frac{1}{\rho} \phi' = \lim_{z \to G^+} \frac{1}{\rho} \phi', \quad \lim_{z \to G^-} \phi = \lim_{z \to G^+} \phi \\
\phi(0) = 0, \quad \phi(D) = 0
\end{aligned}
\]

(3)

where

\[
s(z) = \begin{cases} 
1, & 0 < z \leq H \\
1 + i\sigma(z), & H < z < D.
\end{cases}
\]

The technique described above results in an eigenvalue problem (3), whose solutions satisfy \(\phi \in L^2[0,D]\). So far, we have handled the boundary of infinite region. In the next part, the DTMM for water layer with varying sound speed will be used, so that the mode relation can be derived directly.

### 2.2 DTMM for water layer

For uniform Pekeris waveguide, in which \(\kappa_1\) is constant, TMM is sufficient to derive mode relation. Denoting \(i = \sqrt{-1}, \gamma_0 = \sqrt{\kappa_1^2 - \lambda}, \gamma_2 = \sqrt{\kappa_2^2 - \lambda}\), the solutions of (3) for uniform Pekeris waveguide have the form

\[
\phi(z) = \begin{cases} 
A_0 e^{-i\gamma_0 z} + B_0 e^{i\gamma_0 z}, & 0 \leq z \leq G \\
C_1 e^{-i\gamma_2 z} + C_2 e^{i\gamma_2 z}, & G < z \leq D
\end{cases}
\]

(4)
where $\tilde{z} = z + i \int_0^z \sigma(t) dt$. Using the boundary condition $\phi(0) = 0$, $B_0 = -A_0$ can be solved. To simplify, assume $A_0 = 1$ and $B_0 = -1$ is reasonable. The basic idea is as follows. Firstly, the coefficients $C_1$ and $C_2$ are calculated by applying the TMM, where the transfer matrix can be obtained by the interface conditions. Then, let the second boundary condition $\phi(D) = 0$ be satisfied, the resulting equation is called mode relation, whose roots are corresponding to the eigenvalues of (3).

On the other hand, when $\kappa_1(z)$ varies continuously, TMM is not available, DTMM is a better choice [Zhu and Shen (2011)]. Since the solution of (3) on $z = G^-$ can be written as

$$\phi(G) = A_G e^{-i\gamma G} + B_G e^{i\gamma G},$$

in which $\gamma = \sqrt{\kappa_1(G)^2 - \lambda}$ and the coefficients are to be determined. Suppose $Q_{0^+ \to G^-}$ is the transfer matrix from the $z = 0^+$ to $z = G^-$, it can be expressed analytically by using DTMM [Khorasani and Mehrany (2003)]. The result is

$$Q_{0^+ \to G^-} = \exp(M) = \exp\left[\int_0^G U(z) dz\right],$$

where

$$U(z) = i \gamma_0(z) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \gamma_0'(z) \begin{bmatrix} 1 & -e^{2i\gamma_0(z)z} \\ -e^{-2i\gamma_0(z)z} & 1 \end{bmatrix}.$$

(7)

Denoting $U(z) = U_1(z) + U_2(z)$, then

$$Q_{0^+ \to G^-} = \exp\left[\int_0^G U_1(z) dz\right] \cdot \exp\left[\int_0^G U_2(z) dz\right] = Q_1 \cdot Q_2.$$

By the definition of transfer matrices, we have the transfer relation

$$Q_{0^+ \to G^-} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} A_G \\ B_G \end{bmatrix}.$$

(8)

The integration and matrix exponential in (6) make the relation very complicated. In practical ocean situation, the sound speed usually varies gradually, the second term in (7) can be neglected theoretically. In this way, the coefficients in (5) are expressed as

$$\begin{cases} 
A_G \approx e^{i\gamma G - i\int_0^G \gamma_0(z) dz}, \\
B_G \approx -e^{-i\gamma G + i\int_0^G \gamma_0(z) dz}.
\end{cases}$$
Applying the interface conditions at $G$, the coefficients $C_1$ and $C_2$ in (4) are solved as

$$
C_1 = \left[ (i\gamma_2 \rho_1 + i\gamma_1 \rho_2)T^{-1} - (i\gamma_2 \rho_1 - i\gamma_1 \rho_2)T \right] / (2i\gamma_2 \rho_1 e^{-i\gamma G}) ,
$$

and

$$
C_2 = \left[ (i\gamma_2 \rho_1 - i\gamma_1 \rho_2)T^{-1} - (i\gamma_2 \rho_1 + i\gamma_1 \rho_2)T \right] / (2i\gamma_2 \rho_1 e^{i\gamma G}) ,
$$

in which $T = e^{i\int_0^G \gamma(z)dz}$ for simplification. Let $\phi(D) = 0$, the resulting equation is

$$
e^{i\gamma_2 (2G - \hat{D})} + \frac{(i\gamma_2 \rho_1 - i\gamma_1 \rho_2) - (i\gamma_2 \rho_1 + i\gamma_1 \rho_2)T^2}{(i\gamma_2 \rho_1 + i\gamma_1 \rho_2) - (i\gamma_2 \rho_1 - i\gamma_1 \rho_2)T^2} = 0 ,
$$

where

$$
\hat{D} = D + i \int_0^D \sigma(z)dz .
$$

Furthermore, we can also derive the similar solutions to (4) for varying waveguide by applying this useful tool, that is to say, the eigenfunction can be expressed analytically as

$$
\phi(z) = \begin{cases} 
  e^{-i\int_0^z \gamma_0(r)dr} - e^{i\int_0^z \gamma_1(r)dr}, & 0 \leq z \leq G \\
  C_1 e^{-i\gamma_2 z} + C_2 e^{i\gamma_2 z}, & G < z \leq D .
\end{cases}
$$

Eq. (10) is the mode relation for gradually varied waveguide with PML, and it is solved in the following sections.

### 2.3 Eigenmodes Calculation

For solving the roots of mode relation (10), Newton’s iteration is used. As mentioned above, $\gamma_1$ and $\gamma_2$ are functions of $\lambda$. For PML modes, denote

$$
F(\lambda) = e^{i\gamma_2 (\lambda) (2G - \hat{D})} + \frac{[i\gamma_2 (\lambda) \rho_1 - i\gamma_1 (\lambda) \rho_2] - [i\gamma_2 (\lambda) \rho_1 + i\gamma_1 (\lambda) \rho_2]T^2}{[i\gamma_2 (\lambda) \rho_1 + i\gamma_1 (\lambda) \rho_2] - [i\gamma_2 (\lambda) \rho_1 - i\gamma_1 (\lambda) \rho_2]T^2} .
$$

The iteration procedure is as follows

$$
\lambda_m^{(k+1)} = \lambda_m^{(k)} - \frac{F(\lambda_m^{(k)})}{F'(\lambda_m^{(k)})} ,
$$

in which $m$ is the index of mode.

In order to get more accurate roots, the choices of initial guesses are very important. In this part, we use Chebyshev differentiation matrix to approximate the differential operator [Trefethen (2000)], and the eigenvalues of this matrix are used...
as the initial guesses. In the procedure of approximating the partial differentiation operators, the precision is generally associated with the interpolation nodes. In Chebyshev pseudospectral method, global nodes are used, so the precision is guaranteed, which is the greatest advantage of this method. Once the differential matrix is derived, denoted as \( A_{\text{cheb}} \), the eigenvalues can be easily obtained. For the implementation, there is a already made package \textit{Chebfun} [Driscoll, Bornemann, and Trefethen (2008)], with which the coding is very convenient.

A brief description of the Chebyshev pseudospectral method for the approximation of \( \partial_z^2 + \kappa^2 \) is presented. Since the first and second order Chebyshev differentiation matrices of interval \([-1, 1]\) are given by \( \tilde{D}_N^{(j)} \), \( j = 1, 2 \), in which \( N + 1 \) Chebyshev points as \( \tilde{z}_j = \cos(j\pi/N), j = 0, 1, ..., N \) are used. Using coordinate transformation \( z_j = D + \frac{D}{2}(\tilde{z}_j - 1) \), we have \( D_N^{(j)} = (\frac{2}{D})j\tilde{D}_N^{(j)}, j = 1, 2 \). So for the eigenvalue problem (3), operator \( \partial_z^2 + \kappa^2 \) can be approximated by

\[
A_{\text{cheb}} = D_N^{(2)} + \text{diag}(\kappa(z_j)^2).
\]

Combining with the boundary conditions \( \phi(0) = 0, \phi(D) = 0 \), we have

\[
A_{\text{cheb}} = A_{\text{cheb}}^{\text{old}}(2 : N, 2 : N)
\]

with Matlab notation.

3 Numerical Examples

As above, we have derived the mode relation of infinite domain with varying coefficient by using PML and DTMM, and the methods for initial values and Newton’s iteration are combined for solving the roots of this equation. In this section, we are going to compute the leaky and PML modes by applying the proposed methods in this paper.

As a numerical example, we consider a unbounded waveguide with parameters given by

\[
G = 50\text{m}, \quad \omega = 480, \quad \rho_1 = 1000\text{kg/m}^3, \quad \rho_2 = 1700\text{kg/m}^3, \\
c_1 = 1500e^{-z^2/9}\text{m/s}, \quad c_2 = 1666.67\text{m/s}.
\]

By the definition \( \kappa(z) = \omega/c(z) \), \( \kappa \) is also a varying function. We choose the following parameters for PML:

\[
H = 70\text{m}, \quad D = 80\text{m}, \\
\sigma(z) = \frac{15\tau^3}{1+\tau^2}, \quad \tau = \frac{z-H}{D-H} \quad \text{for } z > H.
\]
Figure 2: Compared the initial solutions with the modes solved by mode relation: the initial solutions are marked by "+", and the modes solved by mode relation are marked by "o".

The discrete modes solved by the above methods are shown in Fig.2. The distribution of all eigenvalues is in the left plot, while the local distribution is in the right one.

Based on the derivation of DTMM, the second term of (7) is neglected. To obtain the exact solutions for comparison, this term should be used, in this way, the exact mode relation for discrete modes can be given as follows:

\[ e^{i \gamma_{2}(2G-2\delta)} + \frac{(a-b)(i\gamma_{2}\rho_{1}-i\gamma_{1}\rho_{2}) + (c-d)(i\gamma_{2}\rho_{1} + i\gamma_{1}\rho_{2})T^{2}}{(a-b)(i\gamma_{2}\rho_{1} + i\gamma_{1}\rho_{2}) + (c-d)(i\gamma_{2}\rho_{1} - i\gamma_{1}\rho_{2})T^{2}} = 0, \]  

in which

\[
\begin{align*}
  & a = Q_{2}(1,1), \\
  & b = Q_{2}(1,2), \\
  & c = Q_{2}(2,1), \\
  & d = Q_{2}(2,2). 
\end{align*}
\]

For another example, choosing \( c_{1} = 1500e^{-(z-25)/300} \text{m/s} \) and keeping the other parameters, the modes are solved as before. The results are described in Figs.4 and 5 and Table 2. In Table 2 the first mode is the propagation mode.
Figure 3: Compared with the exact solutions: the solutions of Eq. (10) are marked by "o", and the exact solutions obtained by Eq. (13) are marked by "*".

Table 1: Sampling for discrete modes with $c_1 = 1500e^{-z^2/9} m/s$.

| k | $\lambda$       | $\lambda_e$       | $|\lambda - \lambda_e|/|\lambda_e|$ |
|---|-----------------|-------------------|----------------------------------|
| 1 | 0.04585+0.00337i| 0.04585+0.00337i  | 1.1500e-07                       |
| 6 | -0.22525+0.00894i | -0.22606+0.00883i | 3.6115e-03                       |
| 12| -1.99171+0.03925i | -1.99171+0.03925i | 1.4611e-06                       |
| 18| -1.76039+0.12657i | -1.76039+0.12657i | 1.3645e-07                       |
| 20| -3.06929+0.24171i | -3.06929+0.24171i | 0                                |
| 25| -5.19328+0.43605i | -5.19328+0.43605i | 0                                |
| 30| -7.86143+0.68626i | -7.86143+0.68626i | 0                                |

When the second term of (7) is used, the exact mode relation (13) is more complicated than (10), and much more difficult to solve. Theoretically, for gradually varied waveguide, the difference can be neglected, and the accurate eigenmodes can be obtained by making the residues of the exact mode relation less than $10^{-6}$. From Figs.2 and 4, it can be seen that the performance of initial guesses are not so good, and more steps of Newton’s iterations are needed to get a better performance.
Figure 4: Compared the initial solutions with the modes solved by mode relation: the initial solutions are marked by "+", and the modes solved by mode relation are marked by "o".

Table 2: Sampling for discrete modes with \( c_1 = 1500e^{-(z-25)/300} \) m/s.

| \( k \) | \( \lambda \)           | \( \lambda_c \)          | \( |\lambda - \lambda_c|/|\lambda_c| \) |
|-------|-----------------|-----------------|------------------|
| 1     | 0.07842+0.00018i | 0.09820+0.00000i | 2.0149e-01       |
| 6     | -0.09133+0.00957i| -0.09134+0.00956i| 1.6371e-04       |
| 12    | -1.71832+0.03783i| -1.71833+0.03783i| 301321e-06       |
| 20    | -3.06902+0.24283i| -3.06902+0.24283i| 9.2898e-07       |
| 25    | -5.19331+0.43719i| -5.19331+0.43719i| 3.1534e-07       |
| 35    | -11.0736+0.99298i| -11.0736+0.99298i| 6.9856e-08       |
| 49    | -27.1419+2.55577i| -27.1419+2.55577i| 1.1004e-08       |

In Figs.3 and 5, the errors will decrease if the index of mode increases, and the roots solved by (10) and (13) are almost the same. This result can be seen more clearly in Tables 1 and 2. In these tables, all modes are sorted from small to large according to their imaginary part, and are denoted as \( \lambda_k \), \( k = 1, 2, \ldots \). We adopt a sampling in accordance to the order of \( k \), it can be found that the mode is very accurate when
its order is large. It illuminates that the mode relation (10) extremely effective for solving large modes. So (13) can be replaced by (10) for large modes in gradually varied waveguide.

4 Conclusions

In this paper, we have presented a way to derive the mode relations for modes in open acoustic waveguide with continuously varying sound speed terminated by PML. Some methods are used, like DTMM, Chebyshev pseudospectral method, and Newton’s iteration. From the numerical examples, the exact mode relation (13) is very complicated, and its solutions are hard to solve. However, the simple formula (10) is a good approximation when the sound speed varies gradually, especially for large modes. The derivation of the relation for varying acoustic waveguide is very meaningful, for it overcomes the difficulty of solving eigenvalue problem (3). The nonlinear equation (10) is easier to be solved by using the simple Newton’s iteration.

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References


