Construction of Operator-Orthogonal Wavelet-Based Elements for Adaptive Analysis of Thin Plate Bending Problems

Y.M. Wang¹,², Q. Wu¹

Abstract: A new kind of operator-orthogonal wavelet-based element is constructed based on the lifting scheme for adaptive analysis of thin plate bending problems. The operators of rectangular and skew thin plate bending problems and the sufficient condition for the operator-orthogonality of multilevel stiffness matrix are derived in the multiresolution finite element space. A new type of operator-orthogonal wavelets for thin plate bending problems is custom designed with high vanishing moments to be orthogonal with the scaling functions with respect to the operators of the problems, which ensures the independent solution of the problems in each scale. An adaptive operator-orthogonal wavelet method is proposed to approximate the exact solution of engineering problems by directly adding wavelets into the local domains until the relative error estimation satisfies the accuracy requirement. Numerical examples demonstrate that the operator-orthogonal method is an accurate and efficient method for bending analysis of thin plate.

Keywords: operator-orthogonal wavelet; thin plate; multiresolution analysis; lifting scheme.

1 Introduction

The wavelets have received an increased attention in the last decades in various engineering disciplines, including signal processing, processing of images, pattern recognition, diagnosing disturbances, mathematical modeling, etc. The generality of their applicability stands directly on the attractive properties, such as periodicity, orthogonality and linear independency [Chui, (1992); Daubechies (1992)]. Current wavelet-based numerical algorithms can be roughly classified as wavelet

¹ School of Automation, Xi’an University of Posts and Telecommunications, Xi’an 710121, People’s Republic of China.
² State Key Laboratory of Acoustics, Institute of Acoustics, Chinese Academy of Sciences, Beijing 100190, China.
Galerkin [Amaratunga and Williams (1994); Mehraeen and Chen (2006)], wavelet
finite element [Chen and Yang (2004); Xiang (2007, 2012)], and wavelet collocation
methods [Bertoluzza and Naldi, (1996); Vasilyev (1995), Libre (2008)], etc. The wavelet
implementation in the finite element analysis, named as wavelet
finite element method, has attracted many researchers in the field of numerical
computation [Dahlke (1997); Cohen (2003); Amaratunga (1994); Dahmen (2001);
Sandeep (2011); Ho (2011); Xiang (2009); Lepik (2005)] and structural analysis [Chen (2004, 2006, 2010); Diaz (2009); He (2012); Mitra (2005); Li, Dong and
Chen (2010,2012); Li and Zhang (2009); Yang (2013)]. Generally, the wavelet
finite elements are constructed by adopting the shape functions to be expressed
in a form of a product of wavelet functions and wavelet coefficients [Ko, Kurdila
and Pilant (1995); Mallat (1999)]. A distinguished feature of wavelet finite element
method is that it combines the versatility of the conventional finite element
method with the accuracy of wavelet functions approximation and various in basis
functions for engineering problems. Diaz constructed Daubechies wavelet finite ele-
ments for beam and plate structures and obtained higher computational accuracy
than traditional finite element analysis [Diaz, Martin and Vampa (2009)]. Zhou
presented a modified Daubechies wavelet approximation for deflections of beams
and square thin plates with both homogeneous and non-homogeneous boundary
conditions based on the modified approximations and Hamilton’s principle [Zhou
and Zhou (2008)]. Pahlavan proposed spectral formulation of finite element meth-
ods using Daubechies compactly-supported wavelets for elastic wave propagation
simulation [Pahlavan (2013)].

Since Daubechies wavelet has no explicit expressions, traditional numerical inte-
grals such as Gauss integral cannot provide desirable precision for the computation
of stiffness matrix [Chen (2004)]. There are various wavelet basis functions with
explicit expressions adopted for the construction of wavelet-based elements, such
as B-spline wavelets, triangle Hermite wavelet, etc. Xiang presented wavelet-based
beam and plate elements using B-spline wavelets on the interval for the bending
and vibration analysis of typical structures such as beam, thin plate, etc [Xiang and
Liang (2011); Xiang, Chen and He (2008); Xiang, Chen and He (2007)]. Han pro-
posed a wavelet-based stochastic finite element method for the bending analysis of
thin plates, which combines the wavelet-based finite element method with Monte
Carlo method [Han, Ren and Huang (2007)]. Zupan proposed spatial triangle Her-
mite wavelet beam element formulation to solve spatial bending and torsion struc-
ture [Zupan, Zupan and Saje (2009)]. Because traditional wavelets are constructed
by the dilation and translation of mother wavelet functions, the characteristics of
wavelet bases are unable to be changed before solving engineering problems, which
results in strong coupling and slow convergence rate in the multiscale computation
of structural analysis.

The emergence of second generation wavelet theory [Sweldens (1997); Sweldens (1996)] has overcome the shortcomings of traditional wavelet-based method. Second generation wavelet is no longer dependent on the telescopic and translation transform, but prediction coefficients and update coefficients to construct wavelet bases flexibly with desired properties, such as compact support, symmetry, high-order vanishing moments. Thus, it provides a great deal of flexibility, and it can be designed according to the properties of the given problem. The second-generation wavelet has gradually being applied to the field of structural analysis. Vasilyev et al. established second-generation wavelet collocation method to solve elliptic and evolution equations over general geometries, such as high-dimensional, spherical domains, etc [Vasilyev (2000, 2003, 2005)]. Wang developed a multiscale lifting algorithm of second-generation wavelet-based finite element method for solving partial differential equations employing the selection of appropriate prediction and update coefficients according to the analyzed problems [Wang, Chen and He (2012)]. Behera presented the multilevel adaptive second generation wavelet collocation method for solving non-divergent barotropic vorticity equation over spherical geodesic grid [Behera (2013)].

In recent years, the generalization of the lifting scheme provides a simple way of constructing biorthogonal wavelet basis functions according to the solution requirements, such as high vanishing moments, high approximation order, symmetry, compact support, etc [Davis (1999); Shui (2004)]. The customization of second generation wavelets in the multiresolution finite element space over general geometries with the objective of developing scale-decoupling algorithms is discussed by Amaratunga, Castrillon, He, etc [Amaratunga and Sudarshan (2006); Castrillón-Candás and Amaratunga (2003); Sudarshan (2006); He, Chen and Xiang (2007)]. Amaratunga presented a framework for the construction of operator-customized wavelets from general finite element interpolation functions, which are scale-orthogonal to the scaling functions at each level with respect to an elliptic partial differential operator [Amaratunga and Sudarshan (2006)]. Castrillon used spatially adaptive lifting wavelets to represent integral operator defined on the three-dimensional geometry, which leads to highly sparse stiffness matrix and less computational time [Castrillón-Candás and Amaratunga (2003)]. D’Heedene constructed decoupling lifting wavelets for arbitrary order Lagrange finite element basis functions on unstructured grid [D’Heedene, Amaratunga and Castrillón-Candás (2005)]. Sudarshan et al. have described a multiresolution modelling with operator-customized wavelets and demonstrated a combined approach for goal-oriented error estimation and adaptivity, where operator-customized wavelets can be constructed from general finite element interpolation functions based on lifting scheme
or Gram–Schmidt orthogonalization [Sudarshan (2006)]. He proposed a new wavelet construction method by designing a suitable prediction operator and update operator according to the requirements of structural analysis [He, Chen and Xiang (2007)]. Quraishi developed a second generation wavelet-based finite element method for solving elliptic PDEs on two dimensional triangulations using customized operator dependent wavelets [Quraishi and Sandeep (2011)]. However, the present wavelets are seldom constructed with user-defined properties especially for multiscale computation of structural problems, such as the operator-orthogonality corresponding to the inner products between scaling functions and wavelets [Wang, Chen and He (2010)].

In this paper, a general construction method of operator-orthogonal wavelet-based elements based on the lifting scheme is presented for adaptive analysis of bending problems of thin plate. An outline of the paper is as follows. Section 2 introduces the multiresolution finite element space. Section 3 discusses the construction of operator-orthogonal wavelet for thin plate analysis according to the operators of the thin plate bending problems. Section 4 presents adaptive scheme for operator-orthogonal wavelet method based on the two-level error estimation. Section 5 demonstrates the numerical performance of the adaptive operator-orthogonal wavelet method and conclusions are drawn in Section 6.

2 Multiresolution finite element space

2.1 Multiresolution analysis

The second generation version of multiresolution analysis (MRA) is an important property in the multilevel approximation of engineering problems. [Sweldens (1997); Sweldens (1996)]. A multiresolution analysis $R$ of $L_2$ is a sequence of closed subspaces $R = \{V_j \subset L_2 | j \in J \subset \mathbb{Z}\}$, such that

1. $V_j \subset V_{j+1}$,
2. $\bigcup_{j \in J} V_j$ is dense in $L_2$,
3. for each $j \in J$, $V_j$ has a Riesz basis given by scaling functions $\{\phi_{j,k} | k \in K(j)\}$, where $j$ is the level of resolution, $J$ is an integer index set associated with resolution levels, $K(j)$ is some index set associated with scaling functions of level $j$, $V_j$ denotes approximation spaces of level $j$. For each $V_j$, there exists a complement of $V_j$ in $V_{j+1}$, namely as $W_j$. Let the spaces $W_j$ be spanned by wavelets, $\psi_{j,m}(x)$ for every $m \in M(j)$, $M(j) = K(j+1) \setminus K(j)$, where $M(j)$ is the difference set of $K(j+1)$ and $K(j)$. Furthermore, let $l \in K(j+1)$ be the index at level $j+1$. 
2.2 Hermite MRA

As stated in Reference [Bathe (1996)], it is possible to construct a valid multiresolution analysis of $V$ provided the interpolating functions are complete and compatible. Based on this premise, the scaling functions of multiresolution finite element space $V_j$ can be chosen as the finite element interpolating functions and the wavelets are the detail interpolating polynomials in the wavelet space $W_j$. A multiresolution decomposition of a finite element space at different levels of resolution is spatial hierarchy:

$$V_j = W_{j-1} \oplus V_{j-1} = W_{j-1} \oplus W_{j-2} \oplus \cdots \oplus W_0 \oplus V_0$$  \hspace{1cm} (1)

Since the finite element spaces are nested, the relation between scaling function $\phi_{j,k}$ and wavelet $\psi_{j,m}$ at level $j$ and $j+1$ satisfies

$$\phi_{j,k} = \sum_l h_{j,k,l} \phi_{j+1,l},$$ \hspace{1cm} (2)

$$\psi_{j,m} = \sum_l g_{j,m,l} \phi_{j+1,l},$$ \hspace{1cm} (3)

where $h_{j,k,l}$ and $g_{j,m,l}$ are referred to as low-pass and high-pass filters, respectively.

A multiresolution analysis allows the approximation of finite energy functions, $u(x) \in L_2(\mathbb{R})$, by a sequence of spaces $V_j$. $u_j(x)$ can be decomposed into its projection on a coarse approximation space $V_0$ along with the projections at multiple levels of wavelet spaces

$$u_j(x) = u_{j-1}(x) + d_{j-1}(x) = u_0(x) + \sum_{i=0}^{j-1} d_i(x) = \sum_l u_{0,l} \phi_{0,l} + \sum_{i=0}^{j-1} \sum_m r_{i,m} \psi_{i,m}$$  \hspace{1cm} (4)

where $u_j(x)$ and $d_j(x)$ are the projections of the function $u(x)$ in the space $V_j$ and $W_j$. $u_{0,l}$ and $r_{i,m}$ are the projection coefficients of $u(x)$ in the space $V_j$ and $W_j$ respectively. Eq.(4) means that the function $u(x)$ can be approximated with the projection $u_j(x)$ in $V_j$ and the projection eventually captures all the details of the initial function $u(x)$ as scale $j$ gets larger (i.e. $j \to \infty$), such as

$$\lim_{j \to \infty} \| u(x) - u_j(x) \| = 0$$  \hspace{1cm} (5)

The larger the scale the lesser the approximating error, so the details will eventually become arbitrarily small, such as

$$\lim_{j \to \infty} d_j(x) = 0$$  \hspace{1cm} (6)
Two-dimensional bicubic Hermite interpolation functions satisfy [Li and Yan (2002); Chien and Shih (2009); Wang (2002)]

\[
\begin{bmatrix}
\phi_{j,k}
\end{bmatrix} = G_{j,l} \begin{bmatrix}
\phi_{j+1,k}
\end{bmatrix}
\] (7)

where

\[
\begin{bmatrix}
\phi_{j,k}
\end{bmatrix} = \begin{bmatrix}
\phi_{j,k}^{(0,0)} & \phi_{j,k}^{(1,0)} & \phi_{j,k}^{(0,1)} & \phi_{j,k}^{(1,1)}
\end{bmatrix}^T
\] (8)

and the nodal degrees of freedom for \(\phi_{j,k}^{(0,0)}, \phi_{j,k}^{(1,0)}, \phi_{j,k}^{(0,1)}, \phi_{j,k}^{(1,1)}\) are the function value, the first partial derivatives and the cross derivative. The coefficient matrix \(G_{j,l}\) in Eq.(7) is determined by the nodal values of scaling functions on the two adjacent scale. Fig.1 shows the refinement relation of bicubic Hermite interpolation.
functions, where the black points denote the scaling functions, the hollow points denote the wavelet functions, the central black points denote scaling functions on the scale $j$ and $j+1$. Fig.2 shows bicubic Hermite scaling functions and wavelets.

3 Operator-orthogonal wavelets for thin plate

3.1 The operators of thin plate

In this section, two kinds of operator-orthogonal wavelets are constructed by the lifting scheme according to the operators of thin plate bending problems in the multiresolution finite element space.

3.1.1 Rectangular thin plate

Fig.3 shows the solving domain $\omega$ of a rectangular thin plate, the side length $l_x$ and $l_y$, respectively.

![Figure 3: Solving domain of rectangular thin plate](image)

The physical equation of thin plate bending problems is

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0 \quad (9)$$

According to Kirchoff plate theory, the generalized function of potential energy for a thin plate is

$$\Pi_p = \frac{1}{2} \iint_{\Omega} \kappa^T D \kappa \, dx \, dy - \iint_{\Omega} w q \, dx \, dy \quad (10)$$

where $w$ is the displacement of the thin plate, $q$ is uniform load, $\kappa$ is generalized strain,

$$\kappa = \left\{ -\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 w}{\partial x \partial y} \right\}^T \quad (11)$$
\( D \) is the elastic matrix in the form

\[
D = D_0 \begin{bmatrix}
1 & \mu & 0 \\
\mu & 1 & 0 \\
0 & 0 & (1-\mu)/2
\end{bmatrix}
\]  

(12)

where \( D_0 = \frac{Et^3}{12(1-\mu^2)} \) is the bending stiffness. \( \mu \) is Poisson’s ratio, \( E \) is Young’s modulus, and \( t \) is the thickness. Applying the principle of minimum of total potential energy, \( \delta \Pi_p = 0 \), we obtain multiscale system of equations for thin plate in terms of Hermite scaling functions and wavelets at level \( j+1 \):

\[
\mathbf{K}_{j+1}\mathbf{u}_{j+1} = \mathbf{P}_{j+1}
\]

(13)

where the stiffness matrix of thin plate on the scale \( j (j \geq 0, j \in \mathbb{Z}) \) can be denoted as

\[
\mathbf{K}_{j+1} = \begin{bmatrix}
K_j(\phi_{j,k}, \phi_{j,k'}) & K_j(\phi_{j,k}, \psi_{j,m}) \\
K_j(\psi_{j,m}, \phi_{j,k}) & K_j(\psi_{j,m}, \psi_{j,m'})
\end{bmatrix}
\]

(14)

where the individual entries in \( \mathbf{K}_{j+1} \) are given as

\[
K_j(\phi_{j,k}, \phi_{j,k'}) = a(\phi_{j,k}, \phi_{j,k'}) \quad \text{(nodal finite element matrix at level } j),
\]

(15)

\[
K_j(\phi_{j,k}, \psi_{j,m}) = a(\phi_{j,k}, \psi_{j,m}) \quad \text{(interaction matrix at level } j),
\]

(16)

\[
K_j(\psi_{j,m}, \phi_{j,k}) = a(\psi_{j,m}, \phi_{j,k}) = K_j(\phi_{j,k}, \psi_{j,m}),
\]

(17)

\[
K_j(\psi_{j,m}, \psi_{j,m'}) = a(\psi_{j,m}, \psi_{j,m'}) \quad \text{(detail matrix at level } j).
\]

(18)

where the node set \( k' \in K(j), m' \in M(j) \).

The stiffness matrix of thin plate in the multiresolution space is

\[
K_j(\phi_{j,k_1}, \phi_{j,k_2}) = D_0 \int_{\Omega} \int \left\{ \frac{\partial^2 \phi_{j,k_1}}{\partial x^2} \frac{\partial^2 \phi_{j,k_2}}{\partial x^2} + \frac{\partial^2 \phi_{j,k_1}}{\partial y^2} \frac{\partial^2 \phi_{j,k_2}}{\partial y^2} + 2(1-\mu) \frac{\partial^2 \phi_{j,k_1}}{\partial x \partial y} \frac{\partial^2 \phi_{j,k_2}}{\partial y} + 2\mu \frac{\partial^2 \phi_{j,k_1}}{\partial x} \frac{\partial^2 \phi_{j,k_2}}{\partial y^2} \right\} dxdy
\]

(19)

The distributed forces \( \mathbf{P}_{j+1} \) and lump forces \( \hat{\mathbf{P}}_{j+1} \) on the scale \( j \) are

\[
\mathbf{P}_{j+1} = \int_{\Omega} q(x) \phi_{j+1} dxdy
\]

(20)

\[
\hat{\mathbf{P}}_{j+1} = \sum_{j+1} P_{j+1} \phi_{j+1}
\]

(21)
The operator of thin plate bending problems can be derived as
\[
a(\psi_{j,m}, \phi_{j,k}) = D_0 \int \int_{\Omega} \left\{ \frac{\partial^2 \psi_{j,m}}{\partial x^2} \frac{\partial^2 \phi_{j,k}}{\partial x^2} + \frac{\partial^2 \psi_{j,m}}{\partial y^2} \frac{\partial^2 \phi_{j,k}}{\partial y^2} + 2(1 - \mu) \frac{\partial^2 \psi_{j,m}}{\partial x \partial y} \frac{\partial^2 \phi_{j,k}}{\partial x \partial y} + 2\mu \frac{\partial^2 \psi_{j,m}}{\partial x^2} \frac{\partial^2 \phi_{j,k}}{\partial y^2} \right\} \mathrm{d}x \mathrm{d}y
\]  
(22)

It is much desirable that the multilevel stiffness matrix are operator-orthogonal, which means that the details do not have any influence on the coarser solution and the engineering problems can be solved on different scales independent of each other. The sufficient condition for the operator-orthogonality of multilevel stiffness matrix is to construct new wavelets orthogonal with respect to the operators of the engineering problems in the multiresolution finite element space.

\[
K_j(\psi_{j,m}, \phi_{j,k}) = a(\psi_{j,m}, \phi_{j,k}) = 0
\]  
(23)

According to Eqs. (2) and (3), the scaling functions and wavelet at a certain level \( j \) can be represented as a linear combination of scaling functions on the finer level \( j + 1 \). Therefore, the operator-orthogonality in Eq. (23) at level \( j \) ensures that the operator-orthogonality at random level \( \tilde{j}(\tilde{j} \in J) \) be satisfied [Amaratunga and Sudarshan (2006)]

\[
a(\psi_{j,m}, \phi_{\tilde{j},k}) = \tilde{H}_{j,k} a(\psi_{j,m}, \phi_{\tilde{j},k}) = 0 \quad (\tilde{k} \in K(\tilde{j}))
\]  
(24)

\[
a(\psi_{\tilde{j},m}, \phi_{j,k}) = \tilde{G}_{j,k} a(\psi_{\tilde{j},m}, \phi_{j,k}) = 0 \quad (\tilde{m} \in M(\tilde{j}))
\]  
(25)

where \( \tilde{H}_{j,k} \) and \( \tilde{G}_{j,k} \) are the low-pass and high-pass filter matrices, respectively.

### 3.1.2 Skew thin plate

Fig.4 shows the solving domain \( \Omega \) of a skew thin plate, \( \alpha \) denotes the skew angle of thin plate.

The oblique coordinate system is constructed when the operator-orthogonal wavelet method is used to solve skew thin plate bending problems. The relationship between the oblique coordinate system \( xoy \) and the Cartesian coordinate system \( \hat{x}\hat{o}\hat{y} \) has the form

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} = \begin{bmatrix}
  1 & -1/\tan \alpha \\
  0 & 1/\sin \alpha
\end{bmatrix} \begin{bmatrix}
  \hat{x} \\
  \hat{y}
\end{bmatrix}
\]  
(26)
The generalized function of potential energy of skew thin plate in the Cartesian coordinate system can be derived as

\[
\Pi_p = \frac{D_0}{2} \iint_{\Omega} \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\mu) \left( \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right) \right\} \, d\hat{x}d\hat{y} - \iint_{\Omega} q(\hat{x},\hat{y})w(\hat{x},\hat{y}) \, d\hat{x}d\hat{y}
\]

According to the principle of minimum of total potential energy, \( \delta \Pi_p = 0 \), the multilevel system of equations for skew thin plate on the scale \( j + 1 \) can be derived as

\[
\bar{K}_{j+1} \bar{u}_{j+1} = \bar{P}_{j+1} \tag{28}
\]

where the distributed forces \( \bar{P}_{j+1} \) and lump forces \( \tilde{P}_{j+1} \) on the scale \( j + 1 \) are

\[
\bar{P}_{j+1} = \sin \alpha \int_{\Omega} q(x)\phi_{j+1} \, dx dy \tag{29}
\]

\[
\tilde{P}_{j+1} = \sin \alpha \sum_{j+1} P_{j+1} \phi_{j+1} \tag{30}
\]

where \( P_{j+1} \) is concentrated loads. The stiffness matrix of skew thin plate in the
multiresolution space is

\[
K_j(\phi_{j,k_1}, \phi_{j,k_2}) = D_0 \int \int_{\Omega_e} \left\{ \frac{\partial^2 \phi_{j,k_1}}{\partial x^2} \frac{\partial^2 \phi_{j,k_2}}{\partial y^2} + \frac{\partial^2 \phi_{j,k_1}}{\partial y^2} \frac{\partial^2 \phi_{j,k_2}}{\partial x^2}
\right.
\]

\[
- 4 \cos \alpha \left( \frac{\partial^2 \phi_{j,k_1}}{\partial x^2} \frac{\partial \phi_{j,k_2}}{\partial x \partial y} + \frac{\partial^2 \phi_{j,k_1}}{\partial y^2} \frac{\partial \phi_{j,k_2}}{\partial x \partial y} \right)
\]

\[
+ 2(1 - \mu \sin^2 \alpha + \cos^2 \alpha) \frac{\partial^2 \phi_{j,k_1}}{\partial x \partial y} \frac{\partial^2 \phi_{j,k_2}}{\partial x \partial y}
\]

\[
+ 2(\mu \sin^2 \alpha + \cos^2 \alpha) \frac{\partial \phi_{j,k_1}}{\partial x} \frac{\partial \phi_{j,k_2}}{\partial x} \right\} \, dx \, dy
\]

(31)

The operator of skew thin plate problems is

\[
a(\psi_{j,m}, \phi_{j,k}) = D_0 \int \int_{\Omega_e} \left\{ \frac{\partial^2 \psi_{j,m}}{\partial x^2} \frac{\partial^2 \phi_{j,k}}{\partial y^2} + \frac{\partial^2 \psi_{j,m}}{\partial y^2} \frac{\partial^2 \phi_{j,k}}{\partial x^2}
\right.
\]

\[
- 4 \cos \alpha \left( \frac{\partial^2 \psi_{j,m}}{\partial x^2} \frac{\partial \phi_{j,k}}{\partial x \partial y} + \frac{\partial^2 \psi_{j,m}}{\partial y^2} \frac{\partial \phi_{j,k}}{\partial x \partial y} \right)
\]

\[
+ 2(1 - \mu \sin^2 \alpha + \cos^2 \alpha) \frac{\partial^2 \psi_{j,m}}{\partial x \partial y} \frac{\partial^2 \phi_{j,k}}{\partial x \partial y}
\]

\[
+ 2(\mu \sin^2 \alpha + \cos^2 \alpha) \frac{\partial \psi_{j,m}}{\partial x} \frac{\partial \phi_{j,k}}{\partial x} \right\} \, dx \, dy
\]

(32)

3.2 Construction of operator-orthogonal wavelets

The lifting scheme proposed by Sweldens is a flexible method for the construction of various new wavelet bases with the desired characteristics. For any multiresolution space, a compactly supported lifting wavelet is built by adding adjacent scaling functions \( \phi_{j,k} \) into the original wavelets \( \psi_{j,m}^{old} \), which is usually selected by the scaling function \( \phi_{j+1,m} \) [Sweldens (1997); Sweldens (1996)]:

\[
\psi_{j,m} = \psi_{j,m}^{old} - \sum_k s_{j,k,m} \phi_{j,k} = \phi_{j+1,m} - \sum_k s_{j,k,m} \phi_{j,k}
\]

(33)

where \( s_{j,k,m} \) are the lifting coefficients. Substituting Eqs. (2) and (3) into Eq. (33), we obtain

\[
\psi_{j,m} = \sum_l g_{j,m,l} \phi_{j+1,l} - \sum_k s_{j,k,m} (\sum_l h_{j,k,l} \phi_{j+1,l}) = H_{j+1,l} \phi_{j+1,l}
\]

(34)

In order to meet the operator-orthogonality in the multiscale computation of engineering problems, the lifting coefficient matrix \( H_{j+1,l} \) can be computed using techniques for computing a basis for the null space of the interaction matrix \( a(\phi_{j,k^*}, \)
\( \phi_{j+1,l} \) as

\[
a(\phi_{j,k*}, \psi_{j,m}) = H_{j+1,l} a(\phi_{j,k*}, \phi_{j+1,l}) = 0
\]

(35)

where \( \phi_{j,k*} \) are all scaling functions on the domains \( \Omega_j \), \( k^* \) are the nodes of scaling functions on \( \Omega_j \). For general engineering problems, the operator-orthogonal wavelet bases can be constructed with \( n + 1 \) vanishing moments with respect to the variables \( x \) and \( y \)

\[
\begin{bmatrix}
a(\phi_{j,k*}, \psi_{j,m}) \\
\frac{a(x^n y^n, \psi_{j,m})}{a(x^n y^n, \phi_{j+1,l})}
\end{bmatrix} = H_{j+1,l} \begin{bmatrix}
a(\phi_{j,k*}, \phi_{j+1,l}) \\
\frac{a(x^n y^n, \phi_{j+1,l})}{a(x^n y^n, \phi_{j+1,l})}
\end{bmatrix} = 0
\]

(36)

The number of the solution of Eq. (36) determines the number of lifted wavelets. The principle of constructing lifted wavelets is to choose proper lifted coefficients from Eq. (36) such that the lifted wavelets are compactly support and the lifting coefficient vectors are linearly independent. Fig.5 shows bicubic Hermite operator-orthogonal wavelets constructed form bicubic Hermite scaling functions with one vanishing moments according to Eq.(36).

![Figure 5: Bicubic Hermite operator-orthogonal wavelets with one vanishing moments](image)

4 Adaptive operator-orthogonal wavelet-based method

4.1 Error analysis

The error estimator of the operator-orthogonal wavelet solution is the key parameter to test the accuracy of operator-orthogonal wavelet method. A two-level error estimator \( \varepsilon_j \) (also called global error estimator) of operator-orthogonal wavelet
method is chosen to be the uniform norm of the difference $e_j$ between the operator-orthogonal wavelet solution $\bar{u}_{j+1}$ and $\bar{u}_j$ at two levels $j+1$ and $j$ respectively in the form

$$
\varepsilon_j = \|e_j\|_\infty = \max |\bar{u}_{j+1} - \bar{u}_j|
$$

(37)

The dimensionless form of global error estimator $\eta_j$ (also called global relative error estimator) on the domain $\Omega_j$ for the operator-orthogonal wavelet method can be defined as

$$
\eta_j = \frac{\max |\bar{u}_{j+1} - \bar{u}_j|}{\max |\bar{u}_{j+1}|}
$$

(38)

The local error estimator $\lambda^r_j$ of an operator-orthogonal wavelet solution in any local domain $\Omega^r_j$ ($r = 1, 2, \cdots$ is the number of local domains) is

$$
\lambda^r_j = |\bar{u}_{j+1}^r - \bar{u}_j^r|
$$

(39)

According to the refinement relation in Eq. (4), the operator-orthogonal wavelet solution can be obtained by using all the operator-orthogonal wavelets on the domain $\Omega^r_j$. As the scale becomes larger, it can be ensured that the error estimator becomes small to satisfy a random threshold value.

4.2 Adaptive operator-orthogonal wavelet algorithm

Reference [Wang, Chen and He (2010)] proposed a multiscale operator-orthogonal wavelet method, also called the multiscale refinement, which gradually approximates the exact solution by adding operator-orthogonal wavelets into global solving domain. In order to solve engineering problems efficiently, an adaptive operator-orthogonal wavelet algorithm, also called adaptive refinement, is proposed. The engineering problems can be solved using the proposed method by adding operator-orthogonal wavelets into local domains with error estimators higher than the given threshold. The adaptive operator-orthogonal wavelet algorithm is given below:

Given error tolerance $\tau$, the threshold value for wavelet refinement $\vartheta (0 < \vartheta \leq 1)$, given initial domain $\Omega_0$ at the scale $j = 0$ and the local domains $\Omega^r_j$, the engineering problems can be solved according to the following steps:

1. Calculate initial operator-orthogonal wavelet solution $u$ in the solving domain $\Omega_0$;
2. Calculate global relative error estimate $\eta_j$, if $\eta_j < \tau$, stop the calculation and output the result;
3. Calculate all the local error estimate $\lambda^r_j$ in the local domains $\Omega^r_j$ and determine the maximum local error estimates $\lambda^\max_j = \max(\lambda^r_j)$;
(4) Generate all local domains that satisfy $\lambda_j^r \geq \vartheta \lambda_j^{\text{max}}$, and save a list of local domains $\tilde{\Omega}_j^r$;

(5) Add detail matrices $K_j(\psi_{j,m},\psi_{j,m'})$ into the multi-scale stiffness matrices $K_{j+1}$ in the local domains $\tilde{\Omega}_j^r$ and let $j = j + 1$;

(6) Solve the multiscale operator-orthogonal wavelet equations and update the operator-orthogonal wavelet solutions and solving domains $\Omega_j^r$, go to (2).

The key for the proposed numerical method is the procedures (3) and (4), in which the local domains are selected according to the local error estimate and the threshold value condition. The operator-orthogonality of multilevel stiffness and mass matrices ensures the incremental computation of eigenvalue solution by the adaptive operator-orthogonal wavelet algorithm. Since the steering parameter $\vartheta$ is chosen randomly, an increasing number of operator-orthogonal wavelets can be added into the local domains and the convergence rate of the solution can be adjusted to users’ computational requirements.

5 Numerical examples

In this section, numerical experiments are presented to demonstrate the efficiency and flexibility of adaptive operator-orthogonal wavelet algorithm. As common structural problems [Rao and Chaudhary (1988); Morley (1963); Timoshenko and Woinowsky-Krieger (1959)], rectangular and skew thin plates are solved by multiscale [He, Chen and Xiang (2007); Wang, Chen and He (2010)] and adaptive operator-orthogonal wavelet method, respectively. In the numerical examples, the threshold values for multiscale and adaptive operator-orthogonal wavelet algorithms are set to be equivalent for the comparison of the accuracy and efficiency. We choose a random threshold value of 0.5 in the numerical examples.

Example 1 Bending analysis of square thin plate simply supported on all four sides on all four sides, the parameters are given as: plate length $L$, thickness $t$, singular load $q_0 = qe^{-100[(x/L-0.5)^2+(y/L-0.5)^2]}$, elastic modulus $E$, Poisson’s ratio $\mu$.

The bicubic Hermite operator-orthogonal wavelets shown in Fig.6 are constructed according to the operator-orthogonality in Eq.(18). Fig.7 shows the relative error of the displacements of square thin plate using multiscale and adaptive operator-orthogonal wavelet solution (Abbreviated as multiscale and adaptive wavelet algorithms) with increasing number of levels and degrees of freedoms. It can be seen that the multi-scale and adaptive operator-orthogonal wavelet method has almost the same convergence rate, but the adaptive operator-orthogonal wavelet method approximates the exact solution with fewer degrees of freedom. Fig.8 shows deformation of square plate simply supported on all four sides. The contour plots
Figure 6: Bicubic Hermite wavelets with (a) two (b) two (c) three vanishing moments of the deformed plate along y direction are shown in Fig.9, the bottom dotted line is the deformation of middle line along y direction and the upper line is the simply supported side. Table 1 illustrates the convergence rate of the displacements by multiscale and adaptive operator-orthogonal wavelet solution with respect to number of levels and degrees of freedoms, respectively. The comparison of the central displacement, central moment and torque moment of the corner points obtained by multiscale, adaptive operator-orthogonal wavelet solution and traditional Shell63 element solution (commercial software ANSYS) with 100×100 meshes is shown in Tables 2. It can be seen that the operator-orthogonal wavelet solutions are match well with those of ANSYS and the degrees of freedom (DOFs) of adaptive operator-orthogonal wavelet method are much less than the other methods.

Example 2 Bending analysis of skew thin plate subjected to uniform load, the parameters are given as: plate length $L$, thickness $t$, uniform load $q$, elastic modulus $E$.
Table 1: Operator-orthogonal wavelet solution for the displacements of square plate simply supported on all four sides

<table>
<thead>
<tr>
<th>Space</th>
<th>Multiscale DOFs</th>
<th>Multiscale $\epsilon_j (10^{-2})$ $\frac{w}{100D_0/qL^4}$</th>
<th>$\eta_j$(%)</th>
<th>Adaptive DOFs</th>
<th>Adaptive $\epsilon_j (10^{-2})$ $\frac{w}{100D_0/qL^4}$</th>
<th>$\eta_j$(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>V0(j=0)</td>
<td>36</td>
<td>———</td>
<td>———</td>
<td>36</td>
<td>———</td>
<td>———</td>
</tr>
<tr>
<td>W0(j=0)</td>
<td>64</td>
<td>0.90303</td>
<td>20.4754</td>
<td>32</td>
<td>0.92721</td>
<td>21.1063</td>
</tr>
<tr>
<td>W1(j=1)</td>
<td>224</td>
<td>0.52636</td>
<td>14.2061</td>
<td>40</td>
<td>0.56845</td>
<td>16.2794</td>
</tr>
<tr>
<td>W2(j=2)</td>
<td>832</td>
<td>0.15164</td>
<td>4.6232</td>
<td>156</td>
<td>0.16651</td>
<td>5.1581</td>
</tr>
<tr>
<td>W3(j=3)</td>
<td>3200</td>
<td>0.08288</td>
<td>2.5763</td>
<td>582</td>
<td>0.08770</td>
<td>2.7215</td>
</tr>
</tbody>
</table>

Table 2: Central displacements and moments of square plate simply supported on all four sides

<table>
<thead>
<tr>
<th>Method</th>
<th>$\frac{w}{100D_0/qL^4}$</th>
<th>$\frac{M_x}{10/qL^2}$</th>
<th>$\frac{M_y}{10/qL^2}$</th>
<th>$\frac{M_{xy}}{10/qL^2}$</th>
<th>DOFs</th>
</tr>
</thead>
<tbody>
<tr>
<td>ANSYS</td>
<td>0.033339</td>
<td>0.061465</td>
<td>0.061465</td>
<td>0.019865</td>
<td>61206</td>
</tr>
<tr>
<td>Multiscale operator-orthogonal wavelet</td>
<td>0.033308</td>
<td>0.060239</td>
<td>0.060239</td>
<td>0.019322</td>
<td>66564</td>
</tr>
<tr>
<td>Error(%)</td>
<td>0.0930</td>
<td>1.9946</td>
<td>1.9946</td>
<td>2.7335</td>
<td></td>
</tr>
<tr>
<td>Adaptive operator-orthogonal wavelet</td>
<td>0.033296</td>
<td>0.060173</td>
<td>0.060173</td>
<td>0.019208</td>
<td>5682</td>
</tr>
<tr>
<td>Error(%)</td>
<td>0.1290</td>
<td>2.1020</td>
<td>2.1020</td>
<td>3.3073</td>
<td></td>
</tr>
</tbody>
</table>
Figure 7: Convergence rate of square plate simply supported on all four sides with (a) number of levels (b) degrees of freedom

Figure 8: Deformed plate

$E$, Poisson’s ratio $\mu$, bevel angle $\alpha$.

Fig.10 shows bicubic Hermite operator-orthogonal wavelets satisfying the operator-orthogonality in Eq.(25). Fig.11 illustrates the relative error of the displacements of skew plate using multiscale and adaptive operator-orthogonal wavelet solution with increasing number of levels and degrees of freedoms for the skew plate, respectively. Fig.12 shows deformation of clamped skew plate, which is subjected to uniform load. The contour plots of the deformed plate along $y$ direction are shown in Fig.13, the deformation of middle line along $y$ direction is shown as dotted.
Figure 9: Contour plots along $y$ direction

Figure 10: Bicubic operator-orthogonal Hermite wavelets with (a) two (b) three (c) three vanishing moments
Table 3 illustrates the convergence rate of the displacements of clamped skew plate under $\alpha=45^\circ$ using the multi-scale and adaptive operator-orthogonal wavelet method, respectively. The comparison of the central displacement, central moment and torque moment of the corner points obtained by multiscale, adaptive operator-orthogonal wavelet solution and ANSYS Shell63 element solution with $100 \times 100$ meshes is shown in Tables 4. It can be seen that numerical solution of the problems using three methods has the same convergence rate, but adaptive operator-orthogonal wavelet method approximates the analytic solution with fewer degrees of freedom. Table 5 illustrates the adaptive operator-orthogonal wavelet solution on the scale $j=3$ and the solution in Reference [Rao, 1988; Morley, 1963] of skew plate under different oblique angle. Table 6 shows the adaptive operator-orthogonal wavelet solution on the scale $j=4$ of central displacement and moment and those of the other FEM (Zienkiewicz, 1988) for the skew plate under skew angle $\alpha=60^\circ$. Both the displacement and moment results indicate that the adaptive operator-orthogonal wavelet method has higher accuracy and less meshes. It can be seen that the analyzed problem is computed with much fewer degrees of freedom although the adaptive solution is close to the solution obtained by multi-scale refinement.

![Figure 11: Convergence rate of skew plate under angle $\alpha=45^\circ$ with (a) number of levels (b) degrees of freedom](image)

**Example 3** Bending analysis of skew thin plate simply supported on two parallel sides, fixed on the other two sides, the parameters are given as: plate length $L$, thickness $t$, load $q_0 = q\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)$, elastic modulus $E$, Poisson’s ratio $\mu$, bevel angle $\alpha$.

The construction method of operator-orthogonal wavelets is the same as those in Fig.10. Fig.14 shows the convergence rate of the displacements of skew plate using...
Table 3: Operator-orthogonal wavelet solution for the displacements of clamped skew plate under $\alpha=45^\circ$

<table>
<thead>
<tr>
<th>Space</th>
<th>Multiscale DOFs</th>
<th>$\varepsilon_j (10^{-2})$ $w 100D_0/qL^4$</th>
<th>$\eta_j(%)$</th>
<th>Adaptive DOFs</th>
<th>$\varepsilon_j (10^{-2})$ $w 100D_0/qL^4$</th>
<th>$\eta_j(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>V0(j=0)</td>
<td>36</td>
<td>—</td>
<td>—</td>
<td>36</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>W0(j=0)</td>
<td>64</td>
<td>0.32409</td>
<td>9.0685</td>
<td>56</td>
<td>0.33117</td>
<td>9.2813</td>
</tr>
<tr>
<td>W1(j=1)</td>
<td>224</td>
<td>0.03304</td>
<td>0.9086</td>
<td>184</td>
<td>0.03428</td>
<td>0.9501</td>
</tr>
<tr>
<td>W2(j=2)</td>
<td>832</td>
<td>0.00279</td>
<td>0.0754</td>
<td>366</td>
<td>0.00280</td>
<td>0.0759</td>
</tr>
<tr>
<td>W3(j=3)</td>
<td>3200</td>
<td>0.00020</td>
<td>0.0053</td>
<td>836</td>
<td>0.00020</td>
<td>0.0053</td>
</tr>
</tbody>
</table>

multiscale and adaptive operator-orthogonal wavelet solution with increasing number of levels and degrees of freedoms. Fig.15 shows deformation of skew thin plate
Table 4: Central displacements and moments of clamped skew plate

<table>
<thead>
<tr>
<th>Method</th>
<th>$w \times 100D_0/qL^4$</th>
<th>$M_x/10qL^2$</th>
<th>$M_y/10qL^2$</th>
<th>$M_{xy}/10qL^2$</th>
<th>DOFs</th>
</tr>
</thead>
<tbody>
<tr>
<td>ANSYS</td>
<td>0.037699</td>
<td>0.098505</td>
<td>0.13401</td>
<td>0.034579</td>
<td>61206</td>
</tr>
<tr>
<td>Multiscale operator-orthogonal wavelet</td>
<td>0.037721</td>
<td>0.098987</td>
<td>0.13464</td>
<td>0.034127</td>
<td>66564</td>
</tr>
<tr>
<td>Error(%)</td>
<td>0.0584</td>
<td>0.4893</td>
<td>0.4701</td>
<td>1.3072</td>
<td></td>
</tr>
<tr>
<td>Adaptive operator-orthogonal wavelet</td>
<td>0.037706</td>
<td>0.098853</td>
<td>0.13418</td>
<td>0.033759</td>
<td>6042</td>
</tr>
<tr>
<td>Error(%)</td>
<td>0.0186</td>
<td>0.3533</td>
<td>0.1269</td>
<td>2.3714</td>
<td></td>
</tr>
</tbody>
</table>

Table 5: Central displacement $w \times 1000 \times D_0/qL^4$ of skew plates for simply supported and clamped boundary conditions at all four sides ($j=3$)

<table>
<thead>
<tr>
<th>Skew angle $\alpha$</th>
<th>Simply supported skew plate subjected to uniform load of intensity $q$</th>
<th>Clamped skew plate subjected to uniform load of intensity $q$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(Morley, 1963)</td>
<td>(Morley, 1963)</td>
</tr>
<tr>
<td>90°</td>
<td>4.0624</td>
<td>4.06</td>
</tr>
<tr>
<td>85°</td>
<td>4.0143</td>
<td>4.01</td>
</tr>
<tr>
<td>80°</td>
<td>3.8739</td>
<td>3.87</td>
</tr>
<tr>
<td>75°</td>
<td>3.6317</td>
<td>3.64</td>
</tr>
<tr>
<td>70°</td>
<td>3.3052</td>
<td>–</td>
</tr>
<tr>
<td>60°</td>
<td>2.5525</td>
<td>2.56</td>
</tr>
<tr>
<td>55°</td>
<td>2.1331</td>
<td>2.14</td>
</tr>
<tr>
<td>50°</td>
<td>1.7116</td>
<td>1.72</td>
</tr>
<tr>
<td>45°</td>
<td>1.3108</td>
<td>1.32</td>
</tr>
<tr>
<td>40°</td>
<td>0.9437</td>
<td>0.958</td>
</tr>
<tr>
<td>30°</td>
<td>0.3894</td>
<td>0.406</td>
</tr>
</tbody>
</table>

simply supported on two parallel sides, fixed on the other two sides. The contour plots of the deformed plate along $y$ direction are shown in Fig.16, the bottom dotted line is the maximum deformation along $y$ direction and the upper line is the simply supported side. Table 7 illustrates the convergence rate of the displacements of skew plate under skew angle $\alpha=30^\circ$ by the multi-scale and adaptive operator-orthogonal wavelet method, respectively. The comparison of maximum displacements and moments by multiscale and adaptive operator-orthogonal wavelet-based
Table 6: Comparison of adaptive operator-orthogonal wavelet results ($j=4$) of central displacement and moment of the skew plate under $\alpha=60^\circ$ with those of traditional FEM (Zienkiewicz, 1988)

<table>
<thead>
<tr>
<th>Mesh</th>
<th>DKQ</th>
<th>ACQ</th>
<th>LSL-Q12</th>
<th>MITC4</th>
<th>MiSP4</th>
<th>MiSP4</th>
<th>MMiSP4</th>
<th>DSQ</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 x 8</td>
<td>0.7876</td>
<td>0.7920</td>
<td>0.7918</td>
<td>0.7610</td>
<td>0.7781</td>
<td>0.7604</td>
<td>0.7840</td>
<td></td>
</tr>
<tr>
<td>12 x 12</td>
<td>0.7909</td>
<td>0.7927</td>
<td>0.7927</td>
<td>0.7785</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td></td>
</tr>
<tr>
<td>16 x 16</td>
<td>0.7920</td>
<td>0.7930</td>
<td>–</td>
<td>–</td>
<td>0.7894</td>
<td>0.7832</td>
<td>0.7871</td>
<td></td>
</tr>
<tr>
<td>Exact</td>
<td>0.7945</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Adaptive operator-orthogonal wavelet</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.7914</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Central moment $M_y \times 10/qL^2$

<table>
<thead>
<tr>
<th>Mesh</th>
<th>DKQ</th>
<th>ACQ</th>
<th>LSL-Q12</th>
<th>MITC4</th>
<th>MiSP4</th>
<th>MiSP4</th>
<th>MMiSP4</th>
<th>DSQ</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 x 8</td>
<td>0.9605</td>
<td>0.9990</td>
<td>0.9777</td>
<td>0.9090</td>
<td>0.9423</td>
<td>0.9052</td>
<td>0.9609</td>
<td></td>
</tr>
<tr>
<td>12 x 12</td>
<td>0.9602</td>
<td>0.9777</td>
<td>0.9680</td>
<td>0.9370</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td></td>
</tr>
<tr>
<td>16 x 16</td>
<td>0.9601</td>
<td>0.9700</td>
<td>–</td>
<td>–</td>
<td>0.9567</td>
<td>0.9466</td>
<td>0.9602</td>
<td></td>
</tr>
<tr>
<td>Exact</td>
<td>0.9589</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Adaptive operator-orthogonal wavelet</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.9622</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 14: Convergence rate of skew plate under oblique angle $\alpha=60^\circ$ with (a) number of levels (b) degrees of freedom

solution with Shell63 element solution with $100 \times 100$ meshes is given in Table 8. The adaptive operator-orthogonal wavelet method shows its advantage over the other two other methods in solving skew plate bending problems with less computational cost.
6 Conclusions

Based on the derivation of the operator of thin plate bending problems, the lifting scheme is used to construct operator-orthogonal wavelets to meet operator-orthogonality of thin plate problems. The numerical examples demonstrate that the operator-orthogonal wavelet-based method realizes independent and accurate solution of thin plate problems in each scale, which is a useful tool to deal with high performance computation in structural analysis. Compared with the traditional fi-
Table 7: Operator-orthogonal wavelet solution for the displacements of skew plate under oblique angle $\alpha=30^\circ$

<table>
<thead>
<tr>
<th>Space</th>
<th>Multiscale DOFs</th>
<th>$\varepsilon_j (10^{-3})$ $w 100D_0/qL^4$</th>
<th>$\eta_j(%)$</th>
<th>Adaptive DOFs</th>
<th>$\varepsilon_j (10^{-3})$ $w 100D_0/qL^4$</th>
<th>$\eta_j(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_0(j=0)$</td>
<td>36</td>
<td>--</td>
<td>--</td>
<td>36</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>$W_0(j=0)$</td>
<td>64</td>
<td>0.75039</td>
<td>18.0295</td>
<td>40</td>
<td>0.76390</td>
<td>18.3671</td>
</tr>
<tr>
<td>$W_1(j=1)$</td>
<td>224</td>
<td>0.52387</td>
<td>12.3858</td>
<td>76</td>
<td>0.55323</td>
<td>13.0952</td>
</tr>
<tr>
<td>$W_2(j=2)$</td>
<td>832</td>
<td>0.22428</td>
<td>5.1605</td>
<td>248</td>
<td>0.23156</td>
<td>5.3342</td>
</tr>
<tr>
<td>$W_3(j=3)$</td>
<td>3200</td>
<td>0.08148</td>
<td>1.8665</td>
<td>820</td>
<td>0.08137</td>
<td>1.8659</td>
</tr>
</tbody>
</table>

Table 8: Maximum displacements and moments of skew plate

<table>
<thead>
<tr>
<th>Method</th>
<th>$w 100D_0/qL^4$</th>
<th>$M_x 10/qL^2$</th>
<th>$M_y 10/qL^2$</th>
<th>$M_{xy} 10/qL^2$</th>
<th>DOFs</th>
</tr>
</thead>
<tbody>
<tr>
<td>ANSYS</td>
<td>0.0043764</td>
<td>0.031042</td>
<td>0.057672</td>
<td>0.023103</td>
<td>61206</td>
</tr>
<tr>
<td>Multiscale</td>
<td>0.0043773</td>
<td>0.031353</td>
<td>0.058538</td>
<td>0.022591</td>
<td>66564</td>
</tr>
<tr>
<td>Error(%)</td>
<td>0.0206</td>
<td>1.0019</td>
<td>1.5016</td>
<td>2.2162</td>
<td></td>
</tr>
<tr>
<td>Adaptive</td>
<td>0.0043759</td>
<td>0.031329</td>
<td>0.058405</td>
<td>0.022482</td>
<td>7216</td>
</tr>
<tr>
<td>Error(%)</td>
<td>0.0114</td>
<td>0.9246</td>
<td>1.2710</td>
<td>2.6880</td>
<td></td>
</tr>
</tbody>
</table>

The finite element method, the adaptive operator-orthogonal wavelet method uses less degrees of freedom to approximate the exact solution of engineering problems. It also shows that operator-orthogonal wavelets bases are attractive for multiscale computation. The advantage of the proposed method over traditional finite element method is that it adds the operator-orthogonal wavelets into the local domains based on two-level error estimation until the solution error satisfies the accuracy requirement. It is promising that the proposed method can be extended to three-dimensional or general structural analysis.

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References


