Three-Variable Shifted Jacobi Polynomials Approach for Numerically Solving Three-Dimensional Multi-Term Fractional-Order PDEs with Variable Coefficients

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Abstract: In this paper, the three-variable shifted Jacobi operational matrix of fractional derivatives is used together with the collocation method for numerical solution of three-dimensional multi-term fractional-order PDEs with variable coefficients. The main characteristic behind this approach is that it reduces such problems to those of solving a system of algebraic equations which greatly simplifying the problem. The approximate solutions of nonlinear fractional PDEs with variable coefficients thus obtained by three-variable shifted Jacobi polynomials are compared with the exact solutions. Furthermore, some theorems and lemmas are introduced to verify the convergence results of our algorithm. Lastly, several numerical examples are presented to test the superiority and efficiency of the proposed method.

Keywords: Three-variable shifted Jacobi polynomials, multi-term fractional-order PDEs, variable coefficients, numerical solution, convergence analysis.

1 Introduction
The elliptic partial differential equations have been applied in various fields of engineering and science. Many important phenomena in electromagnetics, viscoelasticity, fluid mechanics, electrochemistry, biological population models, signals processing [Arpaci (1984); Arpaci and Roache (1972); Myint-U and Debnath (2007); Spotz and Carey (1996); Wang, Zhong and Zhang (2006); Cebeci (2002)] can be well described by elliptic fractional differential equations. For that reason we need a reliable and efficient technique for the solution of fractional differential equations.

The research of numerical solution is still an important subject. Various numerical methods have been proposed to solve such problems. These methods include meshless methods [Dehghan and Shirzadi (2015); Hu, Li and Cheng (2005)], spline collocation methods [Fairweather, Karageorghis and Maack (2011); Abushama and Bialecki (2008)], finite-difference methods [Britt, Tsyntkov and Turkel (2010); Boisvert (1981); Singer and Turkel (2006)], finite element method [Ciarlet (2002)], Chebyshev polynomials method [Ghimire, ¹ College of Mechanical Engineering, Taiyuan University of Science and Technology, Taiyuan 030024, China.
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In this paper, we consider the three-dimensional multi-term fractional-order PDEs with variable coefficients of the following form using three-variable shifted Jacobi polynomials:

\[
a(x, y, z) \frac{\partial^\alpha u(x, y, z)}{\partial x^\alpha} + b(x, y, z) \frac{\partial^\beta u(x, y, z)}{\partial y^\beta} + c(x, y, z) \frac{\partial^\gamma u(x, y, z)}{\partial z^\gamma} + d(x, y, z) \frac{\partial u(x, y, z)}{\partial x} \\
+e(x, y, z) \frac{\partial u(x, y, z)}{\partial y} + k(x, y, z) \frac{\partial u(x, y, z)}{\partial z} + l(x, y, z) u(x, y, z) = f(x, y, z),
\]

where \( \frac{\partial^\alpha}{\partial x^\alpha} , \frac{\partial^\beta}{\partial y^\beta} , \frac{\partial^\gamma}{\partial z^\gamma} , \frac{\partial}{\partial x} , \frac{\partial}{\partial y} , \frac{\partial}{\partial z} \) denotes the Caputo derivative, \( f(x, y, z) \) is a known function and \( u(x, y, z) \) is the solution function to be determined. Subject to the Dirichlet boundary conditions:

\[
\begin{align*}
u(x, y, 0) &= g(x, y, 0), \quad u(x, y, L_3) = g(x, y, L_3), \\
u(x, 0, z) &= g(x, 0, z), \quad u(x, L_2, z) = g(x, L_2, z), \\
u(0, y, z) &= g(0, y, z), \quad u(L_1, y, z) = g(L_1, y, z).
\end{align*}
\]

The current paper is organized as follows: In next Section, the definitions of fractional calculus and shifted Jacobi polynomials, and function approximation are introduced. The differential operational matrix of one-variable shifted Jacobi polynomials is given in Section 3. In Section 4, the error bound and convergence analysis is investigated through some theorems and lemmas. In Section 5, we utilize the three-variable shifted Jacobi polynomials to solve three-dimensional PDEs with variable coefficients. In Section 6, several numerical examples are illustrated to test the proposed method. Finally, a conclusion is drawn in Section 7.

2 Preliminaries and notations

2.1 The fractional derivative in the Caputo sense

Definition 1. The Riemann-Liouville fractional integral operator of order \( \nu (\nu \geq 0) \) is defined as [Zhao, Huang, Xie et al. (2017)]
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\[ J^\nu f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-\tau)^{v-1} f(\tau) d\tau, \quad v > 0, x > 0. \] (3)

\[ J^0 f(x) = f(x). \]

**Definition 2.** The Caputo fractional derivatives of order \( v \) is defined as Zhao et al. [Zhao, Huang, Xie et al. (2017)]

\[ D^v f(x) = J^{m-v} D^m f(x) = \frac{1}{\Gamma(m-v)} \int_0^x (x-\tau)^{m-v-1} \frac{d^m}{d\tau^m} f(\tau) d\tau, m-1 < v \leq m, x > 0, \] (4)

where \( D^m \) is the classical differential operator of order \( m \).

For the Caputo derivative we have

\[ D^\beta x^\beta = \begin{cases} 0, & \text{for } \beta < v, \\ \Gamma(\beta+1) x^{\beta-v}, & \text{for } \beta \geq v. \end{cases} \] (5)

Recall that for \( v \in \mathbb{N} \), the Caputo differential operator coincides with the usual differential operator of an integer order.

Similar to the integer-order differentiation, the Caputo’s fractional differentiation is a linear operation, i.e.

\[ D^\beta (\lambda f(x) + \mu g(x)) = \lambda D^\beta f(x) + \mu D^\beta g(x), \] (6)

where \( \lambda \) and \( \mu \) are constants.

### 2.2 Jacobi polynomials

The well-known Jacobi polynomials are defined on the interval \([-1, 1]\) and can be generated with the aid of the following recurrence formula [Bhrawy and Zaky (2015)]:

\[ P_i^{(\alpha, \beta)}(t) = \frac{(\alpha + \beta + 2i - 1)\{\alpha^2 - \beta^2 + t(\alpha + \beta + 2i)(\alpha + \beta + 2i - 2)\}}{2i(\alpha + \beta + i)(\alpha + \beta + 2i - 2)} P_{i-1}^{(\alpha, \beta)}(t) \\
- \frac{(\alpha + i - 1)(\beta + i - 1)(\alpha + \beta + 2i)}{i(\alpha + \beta + i)(\alpha + \beta + 2i - 2)} P_{i-2}^{(\alpha, \beta)}(t), \quad i = 2, 3, \ldots, \]

where \( P_0^{(\alpha, \beta)}(t) = 1 \) and \( P_1^{(\alpha, \beta)}(t) = \frac{\alpha + \beta + 2}{2} t + \frac{\alpha - \beta}{2} \).

In order to use these polynomials on the interval \( x \in [0, L] \) we define the so-called shifted Jacobi polynomials by introducing the change of variable \( t = \frac{2x}{L} - 1 \). Let the shifted Jacobi
polynomials \( P_{L_i}^{(\alpha, \beta)} \left( \frac{2x}{L} - 1 \right) \) be denoted by \( P_{L_i}^{(\alpha, \beta)}(x) \). Then \( P_{L_i}^{(\alpha, \beta)}(x) \) can be generated from:

\[
P_{L_i}^{(\alpha, \beta)}(x) = \frac{(\alpha + \beta + 2i - 1) \left( \alpha^2 - \beta^2 + \frac{2x}{L} - 1 \right) (\alpha + \beta + 2i - 2)}{2i(\alpha + \beta + i)(\alpha + \beta + 2i - 2)} P_{L_{i-1}}^{(\alpha, \beta)}(x)
\]

(7)

where \( P_{L_0}^{(\alpha, \beta)}(x) = 1 \) and \( P_{L_i}^{(\alpha, \beta)}(x) = \frac{\alpha + \beta + 2}{2} \left( \frac{2x}{L} - 1 \right) + \frac{\alpha - \beta}{2} \).

The analytical form of the shifted Jacobi polynomials \( P_{L_i}^{(\alpha, \beta)}(x) \) of degree \( i \) is given by

\[
P_{L_i}^{(\alpha, \beta)}(x) = \sum_{k=0}^{i} (-1)^{i-k} \frac{\Gamma(i + \beta + 1) \Gamma(i + k + \alpha + \beta + 1)}{\Gamma(k + \beta + 1) \Gamma(i + \alpha + \beta + 1)(i-k)!} \frac{\alpha + \beta + i + 1}{(i+1)!} x^k,
\]

(8)

where \( P_{L_i}^{(\alpha, \beta)}(0) = (-1)^i \frac{\Gamma(i + \beta + 1)}{\Gamma(\beta + 1)i!} \), \( P_{L_i}^{(\alpha, \beta)}(L) = \frac{\Gamma(i + \alpha + 1)}{\Gamma(\alpha + 1)i!} \).

The orthogonality condition of shifted Jacobi polynomials is

\[
\int_0^L P_{L,j}^{(\alpha, \beta)}(x) P_{L,k}^{(\alpha, \beta)}(x) w_L^{(\alpha, \beta)}(x) dx = h_k,
\]

(9)

where \( w_L^{(\alpha, \beta)}(x) = x^\beta (L-x)^\alpha \) and \( h_k = \frac{L^{\alpha+\beta+1} \Gamma(k+\alpha+1) \Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k! \Gamma(k+\alpha+\beta+1)} \), \( i = j \), \( 0, \) \( i \neq j \).

**Definition 3.** Suppose that \( \left\{ P_{L_i}^{(\alpha, \beta)}(x) \right\}_{n=0}^{\infty} \) is the sequence of one-variable shifted Jacobi polynomials on the interval \([0, L]\). Three-variable Jacobi polynomials, \( \left\{ P_{ijk}^{(\alpha, \beta)}(x) \right\}_{i,j,k=0}^{\infty} \), are defined on the domain \( \Omega = [0, L_1] \times [0, L_2] \times [0, L_3] \) as follows:

\[
P_{ijk}^{(\alpha, \beta)}(x, y, z) = P_{L_{i,j}}^{(\alpha, \beta)}(x) P_{L_{j,k}}^{(\alpha, \beta)}(y) P_{L_{i,k}}^{(\alpha, \beta)}(z), i, j, k = 0, 1, 2, \ldots, (x, y, z) \in \Omega.
\]

(10)

**Theorem 1.** The polynomials \( P_{ijk}^{(\alpha, \beta)}(x, y, z) \) are orthogonal with respect to the weight function \( W_{ijk}^{(\alpha, \beta)}(x, y, z) = w_{L_{i,j}}^{(\alpha, \beta)}(x) w_{L_{j,k}}^{(\alpha, \beta)}(y) w_{L_{i,k}}^{(\alpha, \beta)}(z) \) in the domain \( \Omega = [0, L_1] \times [0, L_2] \times [0, L_3] \).

On the hand, the following property is held:

\[
\int_0^1 \int_0^1 \int_0^1 P_{ijk}^{(\alpha, \beta)}(x, y, z) W_{ijk}^{(\alpha, \beta)}(x, y, z) dx dy dz = h_{L_{i,j}}^{(\alpha, \beta)}(x) h_{L_{j,k}}^{(\alpha, \beta)}(y) h_{L_{i,k}}^{(\alpha, \beta)}(z) \delta_{ij} \delta_{jk} \delta_{ik}.
\]
Lemma 1. If \( P_j^{(\alpha,\beta)}(x) \) and \( P_k^{(\alpha,\beta)}(x) \) are \( jth \) and \( kth \) shifted Jacobi polynomials, the product of \( P_j^{(\alpha,\beta)}(x) \) and \( P_j^{(\alpha,\beta)}(x) \) are written as

\[
Q_{j+k}^{(\alpha,\beta)}(x) = \sum_{r=0}^{j+k} \lambda_r^{(j,k)} x^r,
\]

where coefficient \( \lambda_r^{(j,k)} \) are determined as follows:

If \( j \geq k \)

\[
r = 0, 1, \ldots, j + k,
\]

if \( r > j \) then

\[
\lambda_r^{(j,k)} = \sum_{l=r-j}^{k} \gamma_{r-l}^j \gamma_l^k,
\]

else

\[
r_1 = \min \{r, k\},
\]

\[
\lambda_r^{(j,k)} = \sum_{l=0}^{r_1} \gamma_{r-l}^j r_1^k,
\]

end

If \( j < k \);

\[
r = 0, 1, \ldots, j + k,
\]

if \( r \leq j \) then

\[
r_1 = \min \{r, j\},
\]

\[
\lambda_r^{(j,k)} = \sum_{l=0}^{r_1} \gamma_{r-l}^j \gamma_l^k,
\]

else

\[
r_2 = \min \{r, k\},
\]

\[
\lambda_r^{(j,k)} = \sum_{l=r-j}^{r_2} \gamma_{r-l}^j r_1^k,
\]

end

Proof. See Borhanifar et al. [Borhanifar and Sadri (2015)].

Lemma 2. If \( P_i^{(\alpha,\beta)}(x) \), \( P_j^{(\alpha,\beta)}(x) \) and \( P_k^{(\alpha,\beta)}(x) \) are \( i \), \( j \) and \( kth \) shifted Jacobi polynomial, then
\[ q_{ik} = \int_{0}^{1} P_{i}^{(\alpha,\beta)}(x) P_{j}^{(\alpha,\beta)}(x) P_{k}^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) \, dx \]
\[ = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} (\frac{1}{h_{i,j}^{(n,l)}})^{i-l} (\frac{1}{h_{j,k}^{(n,l)}})^{j-k} \lambda^{(n,l)}_{n} \Gamma(i+\beta+1) \Gamma(i+l+\alpha+\beta+1) \Gamma(n+l+\alpha+\beta+2) (i-l)! \] (11)

where \( \lambda^{(n,l)}_{n} \) were introduced in Lemma 1.

### 2.3 Function approximation

A three-variable continuous function \( u(x, y, z) \) in the domain \( \Omega = [0, L_1] \times [0, L_2] \times [0, L_3] \) can be expanded in terms of three-variable shifted Jacobi polynomials as

\[ u(x, y, z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_{ijk} P_{ijk}^{(\alpha,\beta)}(x, y, z), \]

where

\[ u_{ijk} = \frac{1}{h_{i,j}^{(n,l)}} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} u(x, y, z) P_{ijk}^{(\alpha,\beta)}(x, y, z) w^{(\alpha,\beta)}(x, y, z) \, dx \, dy \, dz. \]

In practice, the \((N+1)\) truncated series with respect to all three variables \( x, y \) and \( z \) can be used as an approximation for the given function \( u(x, y, z) \)

\[ u(x, y, z) = u_N(x, y, z) = \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} u_{ijk} P_{ijk}^{(\alpha,\beta)}(x, y, z) = \Phi^{T}(x, y, z) U = U^{T} \Phi(x, y, z), \] (12)

where \( U \) and \( \Phi(x, y, z) \) are the unknown coefficients and three-variable Jacobi polynomials vectors are defined as

\[ U = [u_{000}, u_{001}, \ldots, u_{0NN}, u_{010}, u_{011}, \ldots, u_{NN0}, u_{NN1}, \ldots, u_{NNN}]^{T}, \]

\[ \Phi(x, y, z) = [P_{000}^{(\alpha,\beta)}(x, y, z), \ldots, P_{0NN}^{(\alpha,\beta)}(x, y, z), P_{010}^{(\alpha,\beta)}(x, y, z), \ldots, P_{0NN}^{(\alpha,\beta)}(x, y, z), \ldots, P_{NN0}^{(\alpha,\beta)}(x, y, z), \ldots, P_{NNN}^{(\alpha,\beta)}(x, y, z)]^{T}. \] (13)

The following property of the product of two vectors \( \Phi(x, y, z) \) and \( \Phi^{T}(x, y, z) \) is introduced and applied in solving the three-dimensional PDEs with variable coefficients.

\[ \Phi(x, y, z) \Phi^{T}(x, y, z) V = \tilde{V} \Phi(x, y, z), \] (14)

where \( V \) and \( \tilde{V} \) are, respectively, \((N+1)^3 \times 1\) vector and \((N+1)^3 \times (N+1)^3\) operational matrix of product.

**Theorem 2.** The entries of the matrix \( \tilde{V} \), in Eq. (14), are computed as:
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$$\tilde{V}_{m,n,l,m',n',l'} = \frac{1}{h_{m'}^{(\alpha, \beta)} h_{n'}^{(\alpha, \beta)} h_{l'}^{(\alpha, \beta)}} \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} V_{ijk} q_{ijk'} q_{njm'} q_{l'}$$

where $q_{ijk}$ are introduced by Lemma 2.

**Proof.** See Sadri et al. [Sadri, Amini and Cheng (2017)].

3 The differential operational matrix of one-variable shifted Jacobi polynomials vector

**Lemma 3.** The first-order derivative of the vector $\phi(x)$ can be expressed by

$$\frac{d\phi(x)}{dx} = D^{(1)} \phi(x),$$

where $D^{(1)}$ is the $(N+1) \times (N+1)$ operational matrix of derivative given by

$$D^{(1)} = \begin{cases} C_i(i,j), & i > j, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$C_i(i,j) = \frac{L^{\alpha, \beta}((i+\alpha+\beta+1)(i+\alpha+\beta+2))_{i-j-1} \Gamma(j+\alpha+\beta+1)}{(i-j-1)! \Gamma(2j+\alpha+\beta+1)} \times \begin{pmatrix} -i+1+j, i+j+\alpha+\beta+2, j+\alpha+1 \\ j+\alpha+2, 2j+\alpha+\beta+2 \end{pmatrix}_3 F_2.$$

For the proof see Doha et al. [Doha, Bhrawy and Ezz-Eldien (2012)].

**Theorem 3.** Let $\phi(x)$ be one-variable shifted Jacobi polynomials vector and let also $v > 0$, then

$$D^v \phi(x) = D^{(v)} \phi(x),$$

where $D^{(v)}$ is the $(N+1) \times (N+1)$ operational matrix of derivative of order $v$ in the Caputo sense and is defined by:
where

$$\Delta_v(i,j) = \sum_{k=v}^j \delta_{jk}$$

and $\delta_{jk}$ is given by

$$\delta_{jk} = \frac{(-1)^{i-k} L^{\alpha+\beta-i+1} \Gamma(j+\beta+1) \Gamma(i+\beta+1) \Gamma(i+k+\alpha+\beta+1)}{h_i \Gamma(j+\alpha+\beta+1) \Gamma(k+\beta+1) \Gamma(i+\alpha+\beta+1) \Gamma(k-v+1)(i-k)!}$$

$$\times \sum_{l=0}^{j} \frac{(-1)^{i-l} \Gamma(j+l+\alpha+\beta+1) \Gamma(\alpha+1) \Gamma(l+k+\beta-v+1)}{\Gamma(l+\beta+1) \Gamma(l+k+\alpha+\beta-v+2)(j-l)!!}}.$$

Note that in $D^{(v)}$, the first $\lceil v \rceil$ rows, are all zeros.

**Proof.** See Doha et al. [Doha, Bhrawy and Ezz-Eldien (2012)].

### 4 Error bound and convergence analysis

In this section, we show that the given method in the previous sections, is convergent. For our purpose we will need the following definitions and theorems to obtain an error bound for the proposed method in the Jacobi-weighted Sobolev Space.

**Definition 4.** We define

$$F_N = \text{span}\{P^{(\alpha,\beta)}_{mnl}(x,y,z), 0 \leq m,n,l \leq N\},$$

as the finite-dimensional polynomials space.

**Theorem 4.** Suppose that

$$\frac{\partial^N u(x,y,z)}{\partial x^i \partial y^j \partial z^k} \in C(\Omega)(\Omega = [0,1] \times [0,1] \times [0,1]), i+i_j+i_i = i, i = 0,1,\ldots,N.$$ If $u_N(x,y,z)$ is the Jacobi approximate solution to $u(x,y,z)$ from $F_N$ and $\tilde{u}_N(x,y,z)$ is the Taylor series of the $u(x,y,z)$ of order $N$ respect to each variables $x, y$ and $z$, then an error bound can be presented as follows:
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\[
\left\| u(x, y, z) - u_N(x, y, z) \right\|_{W^{(\alpha, \beta)}} \leq \frac{3^{N+1} M}{(N+1)!} \left( B(\beta + 2, \alpha + 1) \right)^{\frac{3}{2}},
\]

where

\[
M = \max_{0 \leq m, n \leq N+1} \left\{ M_{m,n} \right\}, \quad M_{m,n} = \max_{(x, y, z) \in \Omega} \left| \frac{\partial^{N+1} u(x, y, z)}{\partial x^m \partial y^n} \right|,
\]

and \( B(r, s) \) is the well-known Beta function.

**Proof.** See Sadri et al. [Sadri, Amini and Cheng (2017)].

**Definition 5.** To derive approximation results, we introduce the Jacobi-weighted Space:

\[
F^r_{W^{(\alpha, \beta)}}(\Omega) = \left\{ v \mid v \text{ is measurable and } \|v\|_{r, W^{(\alpha, \beta)}} < \infty \right\}, \quad r \in \mathbb{N}, \quad \Omega = [0,1] \times [0,1] \times [0,1],
\]

equipped with the following norm and semi-norm:

\[
\|v\|_{r, W^{(\alpha, \beta)}} = \left( \sum_{k=0}^{r} \|\partial^k_X v\|_{W_0^{(\alpha+k+1, \beta+k+1)}}^2 \right)^{\frac{1}{2}}, \quad X = (x, y, z),
\]

\[
|v|_{r, W^{(\alpha, \beta)}} = \|\partial^r_X v\|_{W_0^{(\alpha+r, \beta+r)}},
\]

where

\[
\partial^l_X v = \frac{\partial^l v}{\partial x^{l_1} \partial y^{l_2} \partial z^{l_3}}, \quad \sum_{i=1}^{3} l_i = l,
\]

\[
W_0^{(\alpha+l, \beta+l)}(x, y, z) = W^{(\alpha+l_1, \beta+l_2)}(x, y, z) W^{(\alpha+l_3, \beta+l_3)}(x, y, z), \quad \sum_{i=1}^{3} l_i = l.
\]

**Theorem 5.** For any \( u \in F^r_{W^{(\alpha, \beta)}}(\Omega) \), \( r \in \mathbb{N} \), and \( 0 \leq \mu \leq r \), the following estimate holds:

\[
\|u - u_N\|_{r, W^{(\alpha, \beta)}} \leq \Lambda \left( N(N + \alpha + \beta) \right)^{\frac{3}{2}(\mu-r)} |\mu|_{r, W^{(\alpha, \beta)}}, \quad (18)
\]

where \( \Lambda \) is a positive constant independent of any function, \( N, \alpha \) and \( \beta \).

**Proof.** See Sadri et al. [Sadri, Amini and Cheng (2017)].

**Remark.** Let \( u \in F^r_{W^{(\alpha, \beta)}}(\Omega) \) and \( u_N \in F^r_N \) be the Jacobi approximation to \( u \). Then, the following estimates hold for all \( u \in F^r_{W^{(\alpha, \beta)}}(\Omega) \),

\[
\|u - u_N\|_{L^2(\Omega)} \leq \|u - u_N\|_{F^r_{W^{(\alpha, \beta)}}(\Omega)}.
\]
5 Numerical implementation

In the section, we use the three-variable shifted Jacobi polynomials to solve three-dimensional fractional-order PDEs with variable coefficients.

Similarity, the functions \( a(x, y, z), b(x, y, z), c(x, y, z), d(x, y, z), e(x, y, z), k(x, y, z) \) and \( l(x, y, z) \) are also approximated by the three-variable shifted Jacobi polynomials as:

\[
a(x, y, z) = \Phi^T (x, y, z) A, b(x, y, z) = \Phi^T (x, y, z) B, c(x, y, z) = \Phi^T (x, y, z) C, \\
d(x, y, z) = \Phi^T (x, y, z) D, e(x, y, z) = \Phi^T (x, y, z) E, k(x, y, z) = \Phi^T (x, y, z) K, \\
l(x, y, z) = \Phi^T (x, y, z) L.
\]

where \( A, B, C, D, E, K \) and \( L \) can be obtained by Eq. (13).

Using Eqs. (12), (14), (15) and (16) we have Sadri et al. [Sadri, Amini and Cheng (2017)]

\[
a(x, y, z) \left( \frac{\partial^\alpha \Phi(x, y, z)}{\partial x^\alpha} \right) = U^T \left( \frac{\partial^\alpha \Phi(x, y, z)}{\partial x^\alpha} \right) A = U^T \left( \frac{\partial \Phi(x, y, z)}{\partial y} \right) A \\
= U^T \left( \Phi(x, y, z) \right) A = U^T \left( D^\alpha \otimes I_1 \right) \Phi(x, y, z) A
\]

\[
= U^T \left( D^\alpha \otimes I_1 \right) A \Phi(x, y, z),
\]

\[
b(x, y, z) \left( \frac{\partial^\alpha \Phi(x, y, z)}{\partial y^\alpha} \right) = U^T \left( \frac{\partial^\alpha \Phi(x, y, z)}{\partial y^\alpha} \right) B = U^T \left( \frac{\partial \Phi(x, y, z)}{\partial y} \right) B \\
= U^T \left( \Phi(x, y, z) \right) B = U^T \left( I_1 \otimes \left( D^\alpha \otimes I_1 \right) \right) \Phi(x, y, z) B
\]

\[
= U^T \left( I_1 \otimes \left( D^\alpha \otimes I_1 \right) \right) B \Phi(x, y, z),
\]

\[
c(x, y, z) \left( \frac{\partial^\alpha \Phi(x, y, z)}{\partial z^\alpha} \right) = U^T \left( \frac{\partial^\alpha \Phi(x, y, z)}{\partial z^\alpha} \right) C = U^T \left( \frac{\partial \Phi(x, y, z)}{\partial z} \right) C \\
= U^T \left( \Phi(x, y, z) \right) C = U^T \left( I_1 \otimes D^\alpha \right) \Phi(x, y, z) C
\]

\[
= U^T \left( I_1 \otimes D^\alpha \right) C \Phi(x, y, z),
\]

\[
d(x, y, z) \left( \frac{\partial^\alpha \Phi(x, y, z)}{\partial x} \right) = U^T \left( \frac{\partial^\alpha \Phi(x, y, z)}{\partial x} \right) D = U^T \left( \frac{\partial \Phi(x, y, z)}{\partial x} \right) D \\
= U^T \left( \Phi(x, y, z) \right) D = U^T \left( D^\alpha \otimes I_1 \right) \Phi(x, y, z) D
\]

\[
= U^T \left( D^\alpha \otimes I_1 \right) D \Phi(x, y, z).
\]
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\[
e(x, y, z) \frac{\partial u(x, y, z)}{\partial y} = U^T \frac{\partial \Phi(x, y, z)}{\partial y} \Phi^T(x, y, z) E = U^T \frac{\partial (\Phi(x) \otimes \Phi(y) \otimes \Phi(z))}{\partial y} \Phi^T(x, y, z) E
\]

\[
= U^T \left\{ \Phi(x) \otimes \left( \left( \frac{d\Phi(y)}{dy} \right) \otimes \Phi(z) \right) \right\} \Phi^T(x, y, z) E = U^T \left[ I_2 \otimes \left( D^{(0)} \otimes I_2 \right) \right] \Phi(x, y, z) \Phi^T(x, y, z) E
\]

\[
= U^T \left[ I_2 \otimes \left( D^{(0)} \otimes I_2 \right) \right] \tilde{E} \Phi(x, y, z),
\]

\[
k(x, y, z) \frac{\partial u(x, y, z)}{\partial z} = U^T \frac{\partial \Phi(x, y, z)}{\partial z} \Phi^T(x, y, z) K = U^T \frac{\partial (\Phi(x) \otimes \Phi(y) \otimes \Phi(z))}{\partial z} \Phi^T(x, y, z) K
\]

\[
= U^T \left\{ \Phi(x) \otimes \left( \Phi(y) \otimes \frac{d\Phi(z)}{dz} \right) \right\} \Phi^T(x, y, z) K = U^T \left[ I_2 \otimes D^{(0)} \right] \Phi(x, y, z) \Phi^T(x, y, z) K
\]

\[
= U^T \left[ I_2 \otimes D^{(0)} \right] \tilde{K} \Phi(x, y, z),
\]

\[
l(x, y, z) u(x, y, z) = U^T \Phi(x, y, z) \Phi^T(x, y, z) L = U^T \tilde{L} \Phi(x, y, z).
\]

where \( I_1 \) and \( I_2 \) are \((N+1)^2 \times (N+1)^2\) and \((N+1) \times (N+1)\) identity matrices, respectively. Substituting Eqs. (20)-(26) into Eq. (1) we get

\[
U^T (D^x \otimes I_1) \tilde{A} \Phi \Phi^T(x, y, z) + U^T \left[ I_2 \otimes \left( D^y \otimes I_2 \right) \right] \tilde{B} \Phi \Phi^T(x, y, z) + U^T \left[ I_2 \otimes D^z \right] \tilde{C} \Phi \Phi^T(x, y, z)
\]

\[
+ U^T \left[ I_1 \otimes D^{(0)} \right] \tilde{D} \Phi \Phi^T(x, y, z) + U^T \left[ L_1 \otimes I_2 \right] \tilde{E} \Phi \Phi^T(x, y, z) + U^T \left[ I_1 \otimes D^{(0)} \right] \tilde{K} \Phi \Phi^T(x, y, z)
\]

\[
+ U^T \tilde{L} \Phi \Phi^T(x, y, z) = f(x, y, z).
\]

For the Dirichlet boundary condition (2) we have

\[
U^T \Phi(x, y, 0) = g(x, y, 0), \quad U^T \Phi(x, y, L_3) = g(x, y, L_3),
\]

\[
U^T \Phi(x, 0, z) = g(x, 0, z), \quad U^T \Phi(x, L_2, z) = g(x, L_2, z),
\]

\[
U^T \Phi(0, y, z) = g(0, y, z), \quad U^T \Phi(L_1, y, z) = g(L_1, y, z).
\]

Eq. (27) together with Eq. (28) constitutes a system of algebraic equations. Then dispersing the unknown variables \( x, y, \) and \( z \) as the following way:

\[
x_i = \frac{L_1 (2i - 1)}{2(N + 1)}, \quad y_j = \frac{L_2 (2j - 1)}{2(N + 1)}, \quad z_l = \frac{L_3 (2l - 1)}{2(N + 1)}, \quad i, j, l = 1, \ldots, N + 1.
\]

Then we have
\[
\begin{align*}
&U^T (D^r \otimes I_1) \tilde{A} \Phi(x, y, z) + U^T \left(I_2 \otimes (D^r \otimes I_1)\right) \tilde{B} \Phi(x, y, z) + U^T \left(I_1 \otimes D^r\right) \tilde{C} \Phi(x, y, z) \\
&+ U^T \left(D^0 \otimes I_1\right) \tilde{A} \Phi(x, y, z) + U^T \left(I_2 \otimes (D^0 \otimes I_1)\right) \tilde{B} \Phi(x, y, z) + U^T \left(I_1 \otimes D^0\right) \tilde{C} \Phi(x, y, z) \\
&+ U^T \tilde{D} \Phi(x, y, z) = f(x, y, z),
\end{align*}
\]

(30)

Solving this system, the unknown coefficient matrix \( U \) can be obtained. Then using Eq. (12), the unknown solution function \( u(x, y, z) \) is found.

6 Numerical experiments

Example 1. Consider the following three-dimensional multi-term fractional-order PDEs with variable coefficients

\[
\begin{align*}
&\frac{\Gamma(1.75)}{2} x^{1.25} \frac{\partial^{1.5} u(x, y, z)}{\partial x^{1.25}} + \frac{\Gamma(1.5)}{2} y^{1.5} \frac{\partial^{1.5} u(x, y, z)}{\partial y^{1.5}} + \frac{\Gamma(1.25)}{2} z^{1.75} \frac{\partial^{1.75} u(x, y, z)}{\partial z^{1.75}} \\
&+ \frac{1}{2} x \frac{\partial u(x, y, z)}{\partial x} + \frac{1}{2} y \frac{\partial u(x, y, z)}{\partial y} + \frac{1}{2} z \frac{\partial u(x, y, z)}{\partial z} + u(x, y, z) = 7x^2y^2z^2,
\end{align*}
\]

(31)

with the Dirichlet boundary conditions: \( u(x, y, 0) = u(x, 0, z) = u(0, y, z) = 0 \), \( u(x, y, 2) = 4x^2y^2, u(x, 2, z) = 4x^2z^2, u(2, y, z) = 4y^2z^2 \). The analytical solution of this problem is \( u(x, y, z) = x^2y^2z^2 \). When \( N = 2, 4 \) and 6, the absolute errors at some values of \( x, y, z \) are shown in Tab. 1. Tab. 1 shows that the absolute errors decrease as \( N \) increases.

<table>
<thead>
<tr>
<th>((x, y, z))</th>
<th>Analy. Sol.</th>
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<th>( N = 4 )</th>
<th>( N = 6 )</th>
</tr>
</thead>
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<tr>
<td>(0,0,0)</td>
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<td>7.37849180e-4</td>
<td>6.27192198e-5</td>
<td>1.82719189e-6</td>
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<tr>
<td>(0.25,0.25,0.25)</td>
<td>0.000244140625</td>
<td>2.37812719e-3</td>
<td>7.38202901e-5</td>
<td>2.18278112e-6</td>
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<tr>
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<td>3.28172089e-3</td>
<td>8.37191898e-5</td>
<td>2.74918109e-6</td>
</tr>
<tr>
<td>(0.75,0.75,0.75)</td>
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<td>3.72719098e-3</td>
<td>9.37192879e-5</td>
<td>2.35181781e-6</td>
</tr>
<tr>
<td>(1,1,1)</td>
<td>1.00000000000</td>
<td>4.28191890e-3</td>
<td>1.36181289e-4</td>
<td>8.37191807e-7</td>
</tr>
<tr>
<td>(1.25,1.25,1.25)</td>
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<td>4.87319018e-3</td>
<td>1.74910909e-4</td>
<td>4.29810190e-6</td>
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<tr>
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<td>5.27192801e-3</td>
<td>1.52612817e-4</td>
<td>4.63817989e-6</td>
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<tr>
<td>(1.75,1.75,1.75)</td>
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<td>6.78319819e-3</td>
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<tr>
<td>(2,2,2)</td>
<td>64.00000000000</td>
<td>6.15271282e-3</td>
<td>2.75918101e-4</td>
<td>4.18728191e-6</td>
</tr>
</tbody>
</table>
**Example 2.** Consider the following three-dimensional fractional-order PDEs with variable coefficients

\[
\begin{align*}
\frac{\partial^\alpha u(x, y, z)}{\partial x^\alpha} + b(x, y, z) & \frac{\partial^\beta u(x, y, z)}{\partial y^\beta} + c(x, y, z) \frac{\partial^\gamma u(x, y, z)}{\partial z^\gamma} + d(x, y, z) \frac{\partial u(x, y, z)}{\partial x} \\
+ e(x, y, z) & \frac{\partial u(x, y, z)}{\partial y} + f(x, y, z) u(x, y, z) = f(x, y, z),
\end{align*}
\]

(x, y, z) ∈ [0,1] × [0,1] × [0,1].

where \( \alpha = \frac{3}{2}, \beta = \frac{7}{4}, \gamma = \frac{5}{3} \), \( a(x, y, z) = \frac{\sqrt{\pi}}{8} x^{3/2} y^{2} z^{2} \), \( b(x, y, z) = \frac{5\Gamma(3/4)}{96} x^{2} y^{9/4} z^{2} \),

\( c(x, y, z) = \frac{\Gamma(7/3)}{6} x^{2} y^{2} z^{5/3} \), \( d(x, y, z) = \frac{1}{3} x y z^{2} \), \( e(x, y, z) = \frac{1}{3} x^{2} y z^{2} \),

\( k(x, y, z) = \frac{1}{3} x^{2} y^{2} z \), \( l(x, y, z) = x y z \)

and \( f(x, y, z) = 6 x^{3} y^{3} z^{3} + \frac{1}{3} x y z \left[ x^{2} (y^{4} + z^{4}) + y^{2} (x^{4} + z^{4}) + z^{2} (x^{4} + y^{4}) \right] + x y z^{2} \left( x^{2} + y^{2} + z^{2} \right) \).

Subject to the Dirichlet boundary conditions: \( u(x, y, 0) = u(x, 0, z) = u(0, y, z) = 0 \), \( u(x, y, 1) = x y (1 + x^{2} + y^{2}), u(x, 1, z) = x z (1 + x^{2} + z^{2}), u(1, y, z) = y z (1 + y^{2} + z^{2}) \).

The analytical solution of this problem is \( u(x, y, z) = x y z (x^{2} + y^{2} + z^{2}) \). Example 2 and Example 3 show that the numerical solutions approximate to the exact solutions as \( N \) becomes bigger.

(i) When \( x = 1/3 \), \( u(x, y, z) = \frac{y z}{3} \left( \frac{1}{9} + y^{2} + z^{2} \right) \). When \( N = 3, 4 \) and \( 5 \), the graphs of the numerical solutions at \( z = 0.3, 0.6 \) and \( 0.9 \) are shown in Fig. 1.

**Figure 1:** The numerical solution \( u\left(\frac{1}{3}, y, z\right) \) at \( z = 0.3, 0.6 \) and \( 0.9 \) when \( N = 3, 4 \) and \( 5 \)
(ii) When $x = 2/3$, $u(x, y, z) = \frac{2yz}{3} \left( \frac{4}{9} + y^2 + z^2 \right)$. When $N = 3, 4$ and $5$, the graphs of the numerical solutions at $z = 0.3, 0.6$ and $0.9$ are shown in Fig. 2.

![Graphs of numerical solutions](image)

**Figure 2:** The numerical solution $u\left(\frac{2}{3}, y, z\right)$ at $z = 0.3, 0.6$ and $0.9$ when $N = 3, 4$ and $5$

**Example 3.** Consider the following three-dimensional second-order PDEs with variable coefficients

$$a(x, y, z) \frac{\partial^2 u(x, y, z)}{\partial x^2} + b(x, y, z) \frac{\partial^2 u(x, y, z)}{\partial y^2} + c(x, y, z) \frac{\partial^2 u(x, y, z)}{\partial z^2} = f(x, y, z),$$

where $a(x, y, z) = yz, b(x, y, z) = xz, c(x, y, z) = xy$ and

$$f(x, y, z) = \sinh(x+1)\sinh(y+1)\sinh(z+1)(xy+xz+yz).$$

With the Dirichlet boundary conditions:

$$u(x, y, 0) = \sinh(1)\sinh(x+1)\sinh(y+1), u(x, 0, z) = \sinh(1)\sinh(x+1)\sinh(z+1),$$

$$u(0, y, z) = \sinh(1)\sinh(y+1)\sinh(z+1), u(x, y, 1) = \sinh(2)\sinh(x+2)\sinh(y+2),$$

$$u(x, 1, z) = \sinh(2)\sinh(x+2)\sinh(z+2), u(1, y, z) = \sinh(2)\sinh(y+2)\sinh(z+2).$$

The analytical solution of this problem is

$$u(x, y, z) = \sinh(x+1)\sinh(y+1)\sinh(z+1).$$

when $z = 0.5$, $u(x, y, z) = \sinh(1.5)\sinh(1+x)\sinh(1+y)$. The graphs of the numerical and analytical solutions when $N = 2, 3$ and $4$ are shown in Figs. 3-6.
Three-Variable Shifted Jacobi Polynomials Approach for Numerically Solving

Figure 3: Analytical solution

Figure 4: Numerical solution with $N = 2$.

Figure 5: Numerical solution with $N = 3$. 
Figure 6: Numerical solution with $N = 4$.

**Example 4.** Consider Eq. (33), we define the 2-norm error as

$$
\left\| \varepsilon(x, y, z) \right\|_2 = \left( \int_0^1 \left[ u_N(x, y, z) - u(x, y, z) \right]^2 dx \right)^{1/2} \equiv \left( \frac{1}{M} \sum_{i=1}^{M} \left[ u_N(x_i, y_j, z_l) - u(x_i, y_j, z_l) \right]^2 \right)^{1/2},
$$

where $u_N(x, y, z)$ and $u(x, y, z)$ are the approximate and exact solutions respectively.

When $N = 2, N = 3$ and $N = 4$, the 2-norm error $\left\| \varepsilon(x, y, z) \right\|_2$ with $M = 21$ at $y = 0.3, z = 0.6$ are shown in Tab. 2. Tab. 2 shows that the numerical precision can achieve $1e-5 \sim 1e-6$ only small series terms are expanded.

**Table 2:** The 2-norm error $\left\| \varepsilon(x, y, z) \right\|_2$ with $N = 2, 3$ and 4

<table>
<thead>
<tr>
<th></th>
<th>$N = 2$</th>
<th>$N = 3$</th>
<th>$N = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\left| \varepsilon(x, y, z) \right|_2$</td>
<td>$2.36181210e-4$</td>
<td>$3.17281919e-5$</td>
<td>$6.18271018e-6$</td>
</tr>
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</table>

**7 Conclusions**

In this article we have studied a numerical scheme to solve three-dimensional multi-term fractional-order PDEs with variable coefficients. Our approach is based on the three-variable shifted Jacobi polynomials and their operational matrices of fractional derivatives together with a set of suitable collocation nodes. The approximation of the solution together with imposing the collocation nodes is utilized to reduce the computation of this problem to some algebraic equations. The numerical results show that our method is convergent as $N$ increases.

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References


