Finite element-based flow simulations using exponential weighting functions

K. Kakuda¹, Y. Maeda¹

Summary
The applications of a finite element scheme to one-dimensional linear advection-diffusion equation, the incompressible Navier-Stokes equations, and compressible Euler system of equations are presented. The mesh-based scheme is the Petrov-Galerkin weak formulation with exponential weighting functions. Some numerical results demonstrate the workability and the validity of the present approach.

**Keywords:** Finite element, Petrov-Galerkin method, exponential weighting functions, CFD.

Introduction
From a simulation-based design’s point of view, computational fluid dynamics (CFD) is now indispensable in the fields of engineering and science. Numerical instabilities have been experienced in the solution of the advection-dominated system of equations in fluid flow [Brooks and Hughes (1982)]. Various robust schemes have been successfully presented in the mesh-based and meshless-based frameworks for CFD.

In our previous work, we have proposed a finite element-based scheme for solving effectively the problems of incompressible viscous fluid flow [Kakuda and Tosaka (1992)]. The scheme is based on the Petrov-Galerkin weak formulation using exponential weighting functions.

The purpose of this paper is to apply the Petrov-Galerkin finite element-based scheme to one-dimensional linear advection-diffusion equation, the incompressible Navier-Stokes equations, and compressible Euler system of equations. The workability and the validity of the present approach are demonstrated through some numerical results.

Linear advection-diffusion equation
Statement of the problem
Let us first consider the one-dimensional advection-diffusion equation in spatial coordinate, x, given by

\[ \mu \varphi_{,x} = k \varphi_{,xx} \]  

¹College of Industrial Technology, Nihon University, Narashino, Chiba 275-8575, Japan
with the adequate boundary conditions, where \( u \) and \( k \) are the given velocity and diffusivity, respectively. Now, we define the flux \( f = u\phi \) in Eq. (1). With this definition, Eq. (1) is given as follows:

\[
f_{x} = k\phi_{xx}
\]  

(2)

**Petrov-Galerkin finite element formulation**

In order to solve the flux in a stable manner, we shall adopt the Petrov-Galerkin finite element formulation using exponential weighting functions [2]. On the other hand, the conventional Galerkin finite element formulation can be applied to solve numerically Eq. (2).

First of all, we start with the weighted integral expression of the flux in a sub-domain \( \Omega_{i} = [x_{i-1}, x_{i}] \) with respect to weighting functions \( M_{\alpha} \). By applying the divergence theorem to the weighted integral expression, we obtain the following integral form:

\[
\int_{\Omega_{i}} M_{\alpha} N_{\beta} dx f_{\beta} - u \int_{\Omega_{i}} M_{\alpha} N_{\beta} dx \phi_{\beta} = 0
\]  

(3)

The weighting function \( M_{\alpha} \) can be chosen as a general solution which satisfies

\[
u M_{\alpha} + \Delta x_{i} \sigma(u) M_{\alpha, x} = 0
\]  

(4)

where \( \Delta x_{i} = x_{i} - x_{i-1} \), and \( \sigma(u) \) denotes some functions described by Yee et al.[3], which is sometimes referred to as the coefficient of numerical viscosity. The solution of Eq. (4) is

\[
M_{\alpha} = e^{-a(x-x_{\alpha})} \quad (\alpha = x_{i-1}, x_{i})
\]  

(5)

where \( a = \frac{u}{\Delta x_{i} \sigma(u)} \).

Here, applying an element-wise mass lumping to the first term of the left-hand side of Eq. (3), and carrying out exactly those integrals in Eq. (3), we have the following finite element equation

\[
\tilde{c}\delta_{\alpha\beta} f_{\beta} = -u H_{\alpha\beta} \phi_{\beta}
\]  

(6)

where \( \delta_{\alpha\beta} \) is the Kroneker’s delta, and

\[
\tilde{c} = e^{-\gamma} - e^{\gamma}, \quad H_{\alpha\beta} = \begin{bmatrix}
(e^{\gamma} + \frac{c}{\gamma}) & -(e^{-\gamma} + \frac{c}{\gamma}) \\
(e^{\gamma} - \frac{c}{\gamma}) & -(e^{-\gamma} - \frac{c}{\gamma})
\end{bmatrix}, \quad \gamma = \frac{u}{2\sigma(u)}
\]  

(7)

From Eq. (6) we can obtain the following numerical flux \( f_{i-1/2} \) in the subdomain \( \Omega_{i} \)

\[
 f_{i-1/2} = f_{i} + \frac{u}{2}[1 + \{ \text{sgn}(\gamma)\coth|\gamma| - \frac{1}{\gamma}\}](\phi_{i-1} - \phi_{i})
\]  

(8)
and similarly in another subdomain $\Omega_{i+1}$, we have

$$f_{i+1/2} = f_i + \frac{u}{2} [-1 + \{\text{sgn}(\gamma) \coth|\gamma| - \frac{1}{\gamma}\}](\varphi_i - \varphi_{i+1})$$

(9)

where $\text{sgn}(\gamma)$ denotes the signum function.

Let us now consider the Galerkin finite element model for the weighted integral form of Eq. (2). After that, we assume a uniform mesh $\Delta x_i = \Delta x$ for simplicity of the formulation. Taking into consideration the continuity of $\varphi_x$ at nodal point $i$, we can obtain the following discrete form:

$$f_{i-1/2} - f_{i+1/2} + \frac{k}{\Delta x}(\varphi_{i-1} - 2\varphi_i + \varphi_{i+1}) = 0$$

(10)

Substituting Eq. (8) and Eq. (9) into Eq. (10) and after some manipulations, we obtain the following finite difference form:

$$\frac{u}{2\Delta x}(\varphi_{i+1} - \varphi_{i-1}) = (k + \tilde{k}) \frac{\varphi_{i-1} - 2\varphi_i + \varphi_{i+1}}{\Delta x^2}$$

(11)

where for any velocity $u$

$$\tilde{k} = \frac{|u|\Delta x}{2} \{\coth|\gamma| - \frac{1}{|\gamma|}\}$$

(12)

There exist some cases for possible choice of $\sigma(u)$ in Eq. (4). Now if we assume $\sigma(u) = |u|/\alpha$ in which an adhoc parameter $\alpha = \Delta x |u|/k$, then $\gamma$ in Eq. (12) is given as

$$\gamma = \frac{u}{2\sigma(u)} = Pe \quad (\equiv \frac{\Delta x u}{2k} : \text{element Peclet number})$$

(13)

Using the element Peclet number $Pe$ as $\gamma$, we reduce Eq. (11) to the following form:

$$\{\text{sgn}(Pe) - \coth|Pe|\} \varphi_{i+1} + 2\coth|Pe|\varphi_i - \{\text{sgn}(Pe) + \coth|Pe|\} \varphi_{i-1} = 0$$

(14)

This equation has the same structure as the SUPG scheme developed by Hughes et al.[1], and it leads to nodally exact solutions for all values of $Pe$.

**Incompressible Navier-Stokes equations**

**Petrov-Galerkin finite element formulation**

The motion of an incompressible viscous fluid flow is governed by the Navier-Stokes equations in dimensionless form. By applying the time splitting technique
to the set of equations, we can split formally the problem into the nonlinear system of advection-diffusion equations and the linear Euler’s system of equations.

It is well known that the conventional Galerkin finite element scheme using coarse meshes leads to spurious oscillatory solutions for flow simulations at high Reynolds number. Therefore, let us now consider the Petrov-Galerkin finite element formulation using exponential weighting functions [2] to the nonlinear advection-diffusion equations with a Reynolds number $Re$. By applying the divergence theorem to the weighted residual form in a subdomain $\Omega_e$ of the whole domain $\Omega$, and after some manipulations, we have the following weak form:

$$
\int_{\Omega_e} \left\{ u_i(\tilde{u}_i, u_i^p) + u_j u_i,j \right\} M_\alpha d\Omega + \int_{\Omega_e} \frac{1}{Re} u_i,j N_{\alpha,j} d\Omega = \int_{\Gamma_e} \tau_i N_\alpha d\Gamma \quad (15)
$$

in which $\tau_i \equiv u_i,j n_j/Re$, $\Gamma_e$ is the boundary on the subdomain, and $M_\alpha$ denotes the weighting functions given by

$$M_\alpha(x) = \sum_{j,i} N_\alpha(x) e^{-a_i (N_\alpha x_i^p - x_i^p)} \quad , \quad a_i = \frac{\alpha_i}{|L_i|} \text{sgn}(v_i) \quad (16)$$

where $N_\alpha$ is the shape function in three dimensions, $v_i$ is the velocity vector averaged in $\Omega_e$, $L_i$ is the reference length for $x_i$-directions, and $\alpha_i$ is the upwinding parameters which control an effect of the upwinding.

**Numerical example**

As a numerical example, we shall consider the flow around a building in Japan. Fig. (1) shows the geometrical configuration and the numerical results at $Re = 7,900$. The number of nodes is 473,964, $\Delta t = 0.01$, and $\alpha_i = 0.2$.

![Figure 1: Geometrical configuration and numerical results](image)

**Compressible Euler system of equations**

**Statement of the problem**

In this section, in order to develop a high-resolution scheme based on the TVD we have an adhoc function $G$. As a result, we obtain the following modified hyperbolic
Finite element-based flow simulations

systems of conservation laws:

\[ W = S^{-1}U, \quad G = hW, \quad F = \Lambda W + \hat{\Sigma}G \]

\[ \tilde{F} = \text{***}, \quad U_t + \tilde{F}_x = 0 \]

where \( \hat{\Sigma} \) is an \( m \times m \) diagonal matrix associated with the limiter functions.

Petrov-Galerkin finite element formulation

As before, the weighted integral form of Eq. (18) in the subdomain \( \Omega_i = [x_{i-1}, x_i] \) is

\[ \int_{\Omega_i} \{ \tilde{F} - \Lambda_{i-1/2} W - \hat{\Sigma}_{i-1/2} G \} M_\alpha dx = 0 \]

in which \( M_\alpha \) is the exponential weighting functions as follows:

\[ M_\alpha = e^{-a_{i-1/2}(x-x_\alpha)} \quad a_{i-1/2} = \frac{\Sigma_{i-1/2} \Lambda_{i-1/2}}{h_{i-1/2}} \]

By calculating the integrals and using the flux lumping technique such as the mass lumping one, we find the solutions of \( \tilde{F}_{i-1/2} \) in \( \Omega_i \) as follows:

\[ \tilde{F}_{i-1/2} = \tilde{F}_{i-1/2} + \hat{\Sigma}_{i-1/2}[G_i + \frac{1}{2}(I + \tilde{\zeta}_{i-1/2})(G_{i-1} - G_i)] \]

and similarly in an adjacent subdomain \( \Omega_{i+1} \)

\[ \tilde{F}_{i+1/2} = \tilde{F}_{i+1/2} + \hat{\Sigma}_{i+1/2}[G_i - \frac{1}{2}(I - \tilde{\zeta}_{i+1/2})(G_i - G_{i+1})] \]

Numerical example

Let us now consider a Riemann problem, namely the shock-tube problem, for the above Euler system of equations in order to demonstrate the workability and the validity of the present approach. The initial data in a field \( \{x|0 < x < 14\} \) is given as follows:

\[ U(x,0) = U_L \quad \text{if} \quad x < 8 = U_R \quad \text{if} \quad x > 8, \quad \text{where} \quad U_L = (0.445, 0.311, 8.928)^T, \quad U_R = (0.5, 0.0, 1.4275)^T. \]

In Fig. (2) we show the numerical results obtained by the finite element scheme with the Harten’s limiter [Yee, Warming and Harten (1985)] and the Roe’s linearization. The calculations were performed with 100 time steps under the CFL restriction of 0.95. The number of elements is 140.

Conclusions

We have presented a finite element-based scheme using exponential weighting functions for solving numerically the system of equations in fluid flow, such as incompressible Navier-Stokes equations and compressible Euler’s equations. The numerical results demonstrated that the present approach was capable of solving in a stable manner the system of equations.
Figure 2: Numerical results

References

