Solving Rolling Contact Problems Using Boundary Element Method and Mathematical Programming Algorithms

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Abstract: In this work an approach to the two-dimensional steady-state rolling contact problem, with and without force transmission, is presented. The problem is solved by the combination of the Boundary Element Method with a formulation of the variational inequalities that govern the problem in the contact area, producing finally a mathematical programming problem. This formulation avoids the direct use of the contact constrains, but it drives to the minimisation of a non-differentiable function, being necessary the use of an specific numerical tool as the modified Newton's method.

keyword: Rolling, Contact, Mathematical Programming, Boundary Element Method.

1 Introduction

The contact with rolling has occupied an important place in the technical literature relative to mechanical engineering problems. Due to the necessity of having design approaches for mechanical elements of very extended use, such as components of rotating machines, bearings, rollers, etc., as well as for wheels and rolling paths, either for cars, transportation and elevation machines, or trains.

The mechanical study of the rolling problems with *creep* has its beginnings in 1926 with the article of Carter (1926), where the solution for the two-dimensional case of cylinders of similar materials is presented. It can be also highlighted the pioneer work of Kalker (1967), and the papers of Bentall and Johnson (1967, 1968) and Nowell and Hills (1987a,b).

The present formulation of rolling problems springs from the works of Kalker (1971, 1975, 1988), summarised in his book Kalker (1990). In the field of the FEM could be cited the work of Singh and Paul (1974), and Batra (1981), as well as that of Padovan and Zeid (1984), and the prominent works of Oden and coworkers in the field of the variational inequalities and FEM, most collected in the book of Kikuchi and Oden (1988).

Although we can find a wide number of papers on contact problems related with BEM, there are only a few relevant works incorporating the rolling, among them the works of Wang and Knothe (1993), Kong and Wang (1995) and Kalker and Van-Raden (1972), could be spotlighted.

Mathematical Programming Techniques (MPT) are very attractive for solving this type of problems because they allow a more general point of view, and they have very valuable numerical tools. The references using MPT to solve the problems of cuasi-static contact are abundant, especially the articles of Klarbring (1986, 1987), Kwak and Lee (1987), Gakwaya and Lambert (1990) and Kong and Wang (1995).

Alart and Curnier (1991) and Conry and Seireg (1971), are the precursors of the use of the Augmented Lagrangian formulation and the modified Newton's method for the resolution. Later Christensen and Klarbring (1998) gathers most of these techniques, comparing the results and convergence obtained solving with several mathematical programming methods, and enlarging the study of the formulation.

The applications of these techniques to the case of contact with rolling are scarce in the literature, being able to mention among others, Kalker's algorithm KOMBI in Kalker (1990) and those presented by the authors [González and Abascal (1998); Abascal and González (1998)].

This work presents a new approach to the steady-state rolling contact problem for two-dimensional elastic bodies, with and without force transmission, using the BEM to compute the elastic influence coefficients of the surfaces in contact, and a formulation of the variational inequalities that govern the contact conditions based on the projection functions and the Augmented variables.

2 The rolling contact problem

The rolling between two cylinders of parallel axes is the scope of this work. The study is formulated from a two-dimensional point of view; in this way, excluding the displacements in the direction of the cylinder axis and its relative spin from the analysis, assimilating the problem to plane strain.

It is assumed that friction between the cylinders obeys the Coulomb law, where normal and tangential tractions in the slip area are related by a constant friction coefficient. Adopting this approach, each pair of points belonging to each one of the cylinders in rolling (A and B in Fig. 1) can be in any one of the following states: separation, adhesion or slip.

The mathematical expressions of the contact conditions, using Kalker (1990) notation, can be classified in two groups: normal or tangential. It is possible to write each condition as a complementary relation or as a variational inequality (both being equivalent expressions).
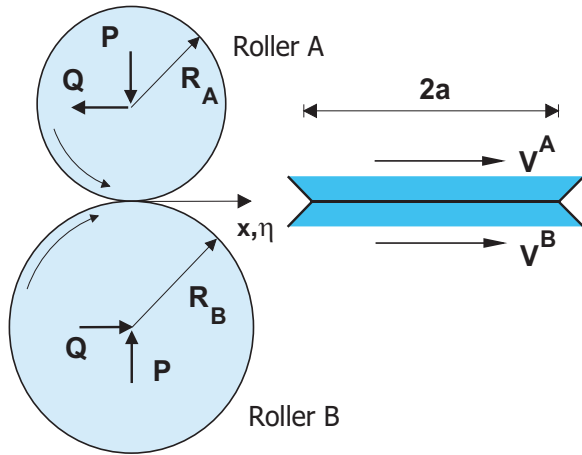
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Figure 1 : Rolling cylinders. Contact zone.

- Normal direction:

complementary relation :
$$\delta_n \geq 0 \quad ; \; p_n \leq 0 \quad ; \; p_n^T \delta_n = 0$$
variational inequality :
$$p_n \in \mathbb{R}^- \quad ; \; \delta_n (p_n - p_n^*) \geq 0 \quad ; \; \forall p_n^* \in \mathbb{R}^- \tag{1}$$

- Tangential direction:

complementary relation :
$$|p_t| \leq g \quad ; \; \lambda \geq 0 \; ; \; s_t = -\lambda p_t \quad ; \; s_t^T (|p_t| - g) = 0$$
variational inequality :
$$p_t \in \mathbb{C}_g \quad ; \; s_t (p_t - p_t^*) \leq 0 \quad ; \; \forall p_t^* \in \mathbb{C}_g \tag{2}$$

where g is the limit of friction and $\mathbb{C}_g$ is a closed interval centred in $\mathbb{R}$ with radius g, $\mathbb{C}_g \equiv \{x \in \mathbb{R} \mid |x| \leq g\}$.

In this way, the possible states in each couple of contact points are characterised by

Sticking points:
$$\begin{cases} p_n^{A,B} \leq 0 & ; \quad p_n^A = p_n^B \quad ; \quad p_t^A = p_t^B \\ \delta_n = 0 & ; \quad s_t = 0 \end{cases} \tag{3}$$

Slipping points:
$$\begin{cases} p_n^{A,B} \leq 0 & ; \quad p_n^A = p_n^B \\ \delta_n = 0 & ; \quad |p_t^{A,B}| = \mu |p_n^{A,B}| \\ sgn(s_t) = -sgn\left(p_t^{A,B}\right) \end{cases} \tag{4}$$

Separation points:
$$\begin{cases} p_n^{A,B} = 0 \quad ; \quad p_t^{A,B} = 0 \quad ; \quad \delta_n \geq 0 \end{cases} \tag{5}$$

where μ is the friction coefficient; p_n^α and p_t^α are the normal and tangential tractions of the contact points of the cylinder

$\alpha = 1, 2$; u_n^α, u_t^α are their respective displacements; δ_n is the normal separation in the deformed state, and s_t is the relative tangential slip velocity.

The separation in the normal direction, δ_n, will be

$$\delta_n = \delta_{no} - \left(u_n^A + u_n^B\right) \tag{6}$$

where the initial separation δ_{no} between the cylinders can be approached geometrically as

$$\delta_{no} = \frac{x^2}{2} \left(\frac{1}{R_A} + \frac{1}{R_B}\right) - \overline{\delta}_{AB} \tag{7}$$

being R_α each one of the cylinder radius; x the eulerian coordinate in the contact area direction which is used to position each pair of points at each time, relative to a rigid body position of the cylinders, and $\overline{\delta}_{AB}$ is the sum of two rigid body displacements externally imposed on each cylinder, usually as an overlap between them.

The relative tangential slip velocity between two point of the cylinders *A* and *B*, is defined as:

$$s_t = \dot{\delta}_t = \frac{d\,\delta_t\,(\eta, \tau)}{d\tau} \tag{8}$$

where τ is the time variable, $\eta = \eta\,(x, \tau)$ is the cartesian coordinate of each point relative to the fixed axes, and δ_t represents the tangential separation, given by

$$\delta_t = \left(\eta^A - \eta^B\right) + \left(u_t^A + u_t^B\right) \tag{9}$$

Substitution into the slip velocity expression, equation (8), yields

$$s_t = \left(V^A - V^B\right) + \left(V^A u_{t,x}^A + V^B u_{t,x}^B\right) + \left(u_{t,\tau}^A + u_{t,\tau}^B\right) \tag{10}$$

where V^α is the rigid body speed in the x direction (coincident with η direction and always negative).

When the steady-state regime is reached, time variations disappear and, like Kalker (1990) shows, the equation (10) can be approximated by

$$s_t = |V| \left[\xi + sgn(V)\left(u_{t,x}^A + u_{t,x}^B\right)\right] \tag{11}$$

where $V = \frac{1}{2}\left(V^A + V^B\right)$ is the mean speed of rolling, and $\xi = \left(V^A - V^B\right)/|V|$ the *creep* or normalised rigid body relative slip velocity.

In this work the boundary displacement derivatives were calculated using a forward finite differences scheme. The speed of the tangential slip for a couple of points i, located at coordinate x_i, under the hypothesis of steady-state rolling, can be expressed as

$$s_{t_i} = |V| \left\{\xi + sgn(V)\left[\left(\frac{u_t^A(x_{i+1}) - u_t^A(x_i)}{h_i}\right) + \left(\frac{u_t^B(x_{i+1}) - u_t^B(x_i)}{h_i}\right)\right]\right\} \tag{12}$$

where h_i denotes the distance between two adjacent points in the rolling direction.

The cylinders were discretized using the classical half-space approach. Having a very small contact zone lenght compared with the cylinder radius, the half-space idealisation is adequate for computing the elastic influence coefficients accurately (Fig. 2), and the normal distance between two contact points should be computed using the equation (7).
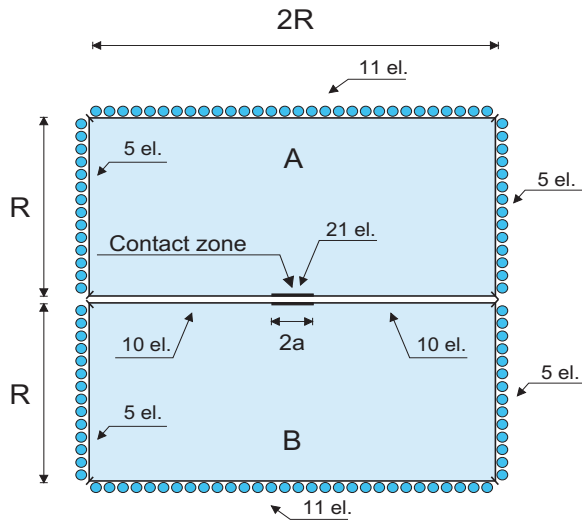

Figure 2 : Approximmating the contact zone by two half-spaces.

This hypothesis, not strictly necessary in the present formulation, is a first approach to the problem of stationary rolling from the BEM point of view and thus permits a comparison ot the results with those existing in the literature based on similar approaches. We should not forget, however, that the proposed methodology allows the study of curved boundaries without additional effort thanks to the numerical power of the BEM. As demonstration of this flexibility the problem of the rolling between a tyred cylinder (with a thin and soft external layer) and other cylinder, will be solved discretizing the thin layer and prescribing the problem boundary conditions, in contrast with the classical methods that need the analytic expressions of the influence coefficients (when they are available).

3   Mathematical formulation of the problem

The rolling contact is a highly non-linear and path-dependent problem; the source of this non-linearity are the boundary conditions used to represent the friction law. In the case of steady-state regime, time dependence disappears and the process is somewhat simpler becoming independent of the solution path.

One of the most used iterative schemes to solve contact problems is denominated PANA in Antes and Panagiotopoulos (1992) that was adopted by Kalker (1988) for the case of 2D

rolling. This process consists on dividing the problem in two chained subproblems NORM and TANG, where their names make reference to the kind of tractions (normal or tangential) wich are the unknowns, while the others are assumed to be equal to those obtained in the last iteration. Because this iteration method might not be valid for complex problems, we present a more general solution procedure. The algorithm we propose, solves the problem in a combined way, without the separation in the normal and tangential part.

The discretization process and the application of the BEM, provide us the elastic equations of each cylinder. After a condensation of the variables not associated with contact areas,

$$\begin{pmatrix} \mathbf{u}_n^\alpha \\ \mathbf{u}_t^\alpha \end{pmatrix} = \begin{bmatrix} \mathbf{S}_{nn}^\alpha & \mathbf{S}_{nt}^\alpha \\ \mathbf{S}_{tn}^\alpha & \mathbf{S}_{tt}^\alpha \end{bmatrix} \begin{pmatrix} \mathbf{p}_n^\alpha \\ \mathbf{p}_t^\alpha \end{pmatrix} + \begin{pmatrix} \mathbf{g}_{ne}^\alpha \\ \mathbf{g}_{te}^\alpha \end{pmatrix} \qquad (13)$$

where $\mathbf{u}_n^\alpha$, $\mathbf{u}_t^\alpha$ and $\mathbf{p}_n^\alpha$, $\mathbf{p}_t^\alpha$ are the vectors that contain the normal and tangential displacements and tractions on the contact area; $\mathbf{g}_{ne}^\alpha$, $\mathbf{g}_{te}^\alpha$ are the normal and tangential displacements associated with the external boundary conditions; $\mathbf{S}_{rs}^\alpha$ is a matrix whose columns represent the contact node displacements in the r-direction due to the application of an unit load in the s-direction at node i, maintaining the rest of the boundary conditions and tractions equal to zero.

Substituting in the equation (6), we obtain one of the departure expressions of the algorithm:

$$\boldsymbol{\delta}_n = \mathbf{d}_n + \mathbf{S}_{nn}\mathbf{p}_n + \mathbf{S}_{nt}\mathbf{p}_t \qquad (14)$$

where

$$\mathbf{d}_n = \boldsymbol{\delta}_{no} - (\mathbf{g}_{ne}^A + \mathbf{g}_{ne}^B) \qquad (15)$$

$$\mathbf{S}_{nn} = -(\mathbf{S}_{nn}^A + \mathbf{S}_{nn}^B) \qquad (16)$$

$$\mathbf{S}_{nt} = -(\mathbf{S}_{nt}^A + \mathbf{S}_{nt}^B) \qquad (17)$$

being

$$\mathbf{p}_n = \mathbf{p}_n^A = \mathbf{p}_n^B \qquad ; \qquad \mathbf{p}_t = \mathbf{p}_t^A = \mathbf{p}_t^B \qquad (18)$$

The matrix form of the slip velocity of each couple of points is obtained applying the $\mathbf{T}$ operator to the tangential displacements vector, providing us with the finite difference approach of its tangential derivative. If equation (12) is organised in matrix form, we can write

$$\mathbf{s}_t = \boldsymbol{\xi} + \mathbf{T}(\mathbf{u}_t^A + \mathbf{u}_t^B) \qquad (19)$$

where the constant $|V|$ has been included in the $\mathbf{s}_t$ vector (it is important to keep in mind that if V is negative the sign of $\mathbf{s}_t$ will not change), and the last equation has been modified to obtain a positive defined matrix $\mathbf{T}$ (the last point of the contact

area, point n, is supposed always in separation, $p_{n_n} = p_{t_n} = 0$), writing the $\mathbf{s}_t$ and $\boldsymbol{\xi}$ vectors in the form:

$$\mathbf{s}_t = \left\{ \begin{array}{c} s_{t_1}/|V| \\ s_{t_2}/|V| \\ \cdots \\ s_{t_{n-1}}/|V| \\ u_{t_n}^A + u_{t_n}^B \end{array} \right\} \quad ; \quad \boldsymbol{\xi} = \left\{ \begin{array}{c} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_{n-1} \\ 0 \end{array} \right\} \tag{20}$$

and the matrix $\mathbf{T}$ as

$$\mathbf{T} = \frac{V}{|V|} \begin{bmatrix} -\frac{1}{h_1} & \frac{1}{h_1} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{h_2} & \frac{1}{h_2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{h_{n-1}} & \frac{1}{h_{n-1}} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{21}$$

Substituting the expression of the tangential displacements in the equation (19) finds that

$$\mathbf{s}_t = \boldsymbol{\xi}^* + \mathbf{B}_{tn}\mathbf{p}_n + \mathbf{B}_{tt}\mathbf{p}_t \tag{22}$$

where

$$\boldsymbol{\xi}^* = \boldsymbol{\xi} + \mathbf{T}(\mathbf{g_{te}^A} + \mathbf{g_{te}^B}) \tag{23}$$

$$\mathbf{B}_{tn} = \mathbf{T}\left(\mathbf{S}_{tn}^A + \mathbf{S}_{tn}^B\right) \tag{24}$$

$$\mathbf{B}_{tt} = \mathbf{T}\left(\mathbf{S}_{tt}^A + \mathbf{S}_{tt}^B\right) \tag{25}$$

In this way, after the resolution of the elastic problem using BEM and later condensation, we have obtained the linear relationship between the normal separation $\boldsymbol{\delta}_n$, the slip velocity $\mathbf{s}_t$ and the normal and tangential tractions in the contact area. Flexibility version of this relationship can be expressed by

$$\left(\begin{array}{c} \boldsymbol{\delta}_n \\ \mathbf{s}_t \end{array} \right) = \left(\begin{array}{c} \mathbf{d}_n \\ \boldsymbol{\xi}^* \end{array} \right) + \left[\begin{array}{cc} \mathbf{S}_{nn} & \mathbf{S}_{nt} \\ \mathbf{B}_{tn} & \mathbf{B}_{tt} \end{array} \right] \left(\begin{array}{c} \mathbf{p}_n \\ \mathbf{p}_t \end{array} \right) \tag{26}$$

This equation together with the contact conditions expressed by the equations (1) and (2) for all the contact point pairs, completes the mathematical description of the problem.

4 Fulfilling the contact constraints

Alart and Curnier (1991) developed a mixed penalty-duality formulation of the frictional contact problem to treat its multivalued aspects, inspired by the Augmented Lagrangian method. They derived an non-symmetrical operator using a *quasi*-Augmented Lagrangian formulation to palliate the absence of a genuine one, exhibiting its properties and establishing a necessary and sufficient condition on the friction coefficient for the solution of curved, discrete, small slip contacts to be unique and to guarantee the convergence of Newton's method.

Inspired by that work, Christensen and Klarbring (1998) proposes a variation of the same formulation, providing a system of B-differentiable equations, that allows him to use the convergence theorems and the powerful Newton's algorithm developed by Pang (1990). Following the same line, in this work we formulated the contact conditions in a similar way, extrapolating them directly to the particular case of rolling contact and using BEM.

4.1 Normal problem

If the tangential tractions are assumed well known ($\mathbf{p}_t = \overline{\mathbf{p}}_t$), the problem could be exclusively considered function of the normal variables, obtaining the denominated *Normal problem*. The possible states for each pair of points according to the normal direction are two: contact and separation. This means that each couple should satisfy the following conditions:

$$\begin{array}{ll} Contact: \\ \quad \delta_{n_i} = 0 \quad ; \; p_{n_i} \leq 0 \quad \Rightarrow p_{n_i} + r\delta_{n_i} \leq 0 \\ Separation: \\ \quad \delta_{n_i} > 0 \quad ; \; p_{n_i} = 0 \quad \Rightarrow p_{n_i} + r\delta_{n_i} > 0 \end{array} \tag{27}$$

being r a penalty parameter, positive and real, coming from the *quasi*-Augmented Lagrangian formulation.

These constraints can be summarised in the vectorial equation

$$\mathbf{p}_n = \min(0, \mathbf{p}_n + r\boldsymbol{\delta}_n) \tag{28}$$

relation that will only be verified if each pair of points fulfil the contact conditions of one of its possible states.

This condition can also be written by means of the expression

$$\min(-\mathbf{p}_n, r\boldsymbol{\delta}_n) = 0 \tag{29}$$

which is equivalent to impose that the normal tractions are always of compression ($-\mathbf{p}_n \geq 0$), that the overlapping between cylinders doesn't exist ($\boldsymbol{\delta}_n \geq 0$), and that at least one of the two variables is zero.

The normal problem can be formulated combining the elastic equations (26) with the contact conditions defined by the ecs. (28) or (29),

$$\boldsymbol{\Phi}_n(\mathbf{s}_n, \mathbf{p}_n) \equiv \left[\begin{array}{c} \boldsymbol{\delta}_n - (\mathbf{d}_n + \mathbf{S}_{nt}\overline{\mathbf{p}}_t) - \mathbf{S}_{nn}\mathbf{p}_n \\ \mathbf{p}_n - \min(0, \mathbf{p}_n + r\boldsymbol{\delta}_n) \end{array} \right] = 0 \tag{30}$$

where the main peculiarity of the system resides in the appearance of non-differentiable points in the space of solutions due to the formulation of the contact constrains.

4.2 Tangential problem

The formulation of the tangential part of the problem is normal tractions dependent. Let us suppose that the distribution of normal tractions is known and therefore we can obtain the

vector $\mathbf{g} = -\mu\overline{\mathbf{p}}_n$. To formulate the contact constrains it is convenient to have previously introduced two definitions: the euclidean distance between a point x and an interval, and the projection function of a variable x over an interval.

Definition 1: The euclidean distance from a point $x \in \mathbb{R}$ to a closed and centred interval $\mathbb{C}_g \subset \mathbb{R}$, is defined as $dist(x, \mathbb{C}_g) = \max(x - g, 0) - \min(x + g, 0)$.

Definition 2: The projection function of a point $x \in \mathbb{R}$ on the closed and centred interval $\mathbb{C}_g \subset \mathbb{R}$, is defined as $\mathbb{P}_{\mathbb{C}_g}(x) = x - sgn(x) \cdot dist(x, \mathbb{C}_g)$.

Using these definitions, the possible states of each contact point for the tangential case will be:

$Adhesion$:
$$s_{t_i} = 0 \quad ; \ |p_{t_i}| \le g_i \quad \Rightarrow |p_{t_i} - rs_{t_i}| \le g_i$$
$Slip$:
$$s_{t_i} \ne 0 \ ; \ p_{t_i} s_{t_i} < 0 \quad ; \ |p_{t_i}| = g_i \quad \Rightarrow |p_{t_i} - rs_{t_i}| > g_i$$
$Separation$:
$$s_{t_i} \ne 0 \ ; \ p_{t_i} = 0 \quad ; \ g_i = 0 \quad \Rightarrow \quad \mathbb{C}_{g_i} = \emptyset$$

being easy to prove that fulfillment of these inequalities can be guaranteed by satisfying

$$p_{t_i} = \mathbb{P}_{\mathbb{C}_{g_i}}(p_{t_i} - rs_{t_i}) \tag{31}$$

The tangential part of the problem could be written organising in a system of equations equation (26) and the contact conditions

$$\boldsymbol{\Phi}_t(\mathbf{s}_t, \mathbf{p}_t) \equiv \begin{bmatrix} \mathbf{s}_t - (\boldsymbol{\xi}^* + \mathbf{B}_{tn}\overline{\mathbf{p}}_n) - \mathbf{B}_{tt}\mathbf{p}_t \\ \mathbf{p}_t - \mathbb{P}_{\mathbb{C}_g}(\mathbf{p}_t - r\mathbf{s}_t) \end{bmatrix} = 0 \tag{32}$$

where, as in the normal problem, the equilibrium is imposed by the first equation, and the contact conditions (applied point by point) are established by the second one. Being the projection function, included in the previous equation, the source of non-differentiability.

4.3 The coupled problem

The solution of the coupled normal-tangential problem is more complicated than each one separately because the normal tractions and thus the friction limit, are unknowns. To formulate the problem we have used the direct combination of the equations (30) and (32), substituting the friction limit by $-\mu\min(0, \mathbf{p}_n + r\boldsymbol{\delta}_n)$, in such a way that it is always possible a projection onto the valid region of the friction cone, taking into account the equation (28). We obtain in this way:

$$\boldsymbol{\Phi}(\boldsymbol{\delta}_n, \mathbf{s}_t, \mathbf{p}_n, \mathbf{p}_t) \equiv$$
$$\begin{bmatrix} \boldsymbol{\delta}_n - \mathbf{d}_n - \mathbf{S}_{nn}\mathbf{p}_n - \mathbf{S}_{nt}\mathbf{p}_t \\ \mathbf{s}_t - \boldsymbol{\xi}^* - \mathbf{B}_{tn}\mathbf{p}_n - \mathbf{B}_{tt}\mathbf{p}_t \\ \mathbf{p}_n - \min(0, \mathbf{p}_n + r\boldsymbol{\delta}_n) \\ \mathbf{p}_t - \mathbb{P}_{\mathbb{C}_{-\mu\min(0, \mathbf{p}_n + r\boldsymbol{\delta}_n)}}(\mathbf{p}_t - r\mathbf{s}_t) \end{bmatrix} = 0 \tag{33}$$

The solution of this non-differentiable equation system will provide us with the solution of the problem.

5 Solution procedure. GNM Algorithm

To solve equation (33) we have used the Generalised Newton's Method (GNM), an extension of the Newton's Method for non-differentiable functions, Pang (1990). In practice, during the iteration process we will try many solutions, and some of them will be near to points where the function is non-differentiable, but it is unusual having solution points situated exactly over one of them. Based on this assumption the jacobian will be obtained ignoring the frontiers between the different contact states where the function is not differentiable. Dividing the system in its differentiable and non-differentiable parts, we have:

$$\boldsymbol{\Phi}(\boldsymbol{\delta}_n, \mathbf{s}_t, \mathbf{p}_n, \mathbf{p}_t) = \boldsymbol{\Phi}^D(\boldsymbol{\delta}_n, \mathbf{s}_t, \mathbf{p}_n, \mathbf{p}_t) + \boldsymbol{\Phi}^{ND}(\boldsymbol{\delta}_n, \mathbf{s}_t, \mathbf{p}_n, \mathbf{p}_t) \tag{34}$$

where the differentiable part is

$$\boldsymbol{\Phi}^D(\boldsymbol{\delta}_n, \mathbf{s}_t, \mathbf{p}_n, \mathbf{p}_t) \equiv \begin{bmatrix} \boldsymbol{\delta}_n - \mathbf{d}_n - \mathbf{S}_{nn}\mathbf{p}_n - \mathbf{S}_{nt}\mathbf{p}_t \\ \mathbf{s}_t - \boldsymbol{\xi}^* - \mathbf{B}_{tn}\mathbf{p}_n - \mathbf{B}_{tt}\mathbf{p}_t \\ \mathbf{p}_n \\ \mathbf{p}_t \end{bmatrix} \tag{35}$$

with the following jacobian

$$\partial\boldsymbol{\Phi}^D = \begin{bmatrix} \mathbf{I} & \mathbf{0} & -\mathbf{S}_{nn} & -\mathbf{S}_{nt} \\ \mathbf{0} & \mathbf{I} & -\mathbf{B}_{tn} & -\mathbf{B}_{tt} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{36}$$

and the non-differentiable part

$$\boldsymbol{\Phi}^{ND}(\boldsymbol{\delta}_n, \mathbf{s}_t, \mathbf{p}_n, \mathbf{p}_t) \equiv$$
$$\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ -\min(0, \mathbf{p}_n + r\boldsymbol{\delta}_n) \\ -\mathbb{P}_{\mathbb{C}_{-\mu\min(0, \mathbf{p}_n + r\boldsymbol{\delta}_n)}}(\mathbf{p}_t - r\mathbf{s}_t) \end{bmatrix} \tag{37}$$

where the jacobian is obtained depending on the contact state of each node i,

1. If $p_{n_i} + r\delta_{n_i} > 0$:
$$[\partial\boldsymbol{\Phi}^{ND}]_i = 0 \tag{38}$$

2. If $p_{n_i} + r\delta_{n_i} < 0$ and $|p_{t_i} - rs_{t_i}| < -\mu(p_{n_i} + r\delta_{n_i})$:
$$[\partial\boldsymbol{\Phi}^{ND}]_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -r & 0 & -1 & 0 \\ 0 & r & 0 & -1 \end{bmatrix} \tag{39}$$

3. And if $p_{n_i} + r\delta_{n_i} < 0$ and $|p_{t_i} - rs_{t_i}| > -\mu(p_{n_i} + r\delta_{n_i})$:
$$[\partial\boldsymbol{\Phi}^{ND}]_i =$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -r & 0 & -1 & 0 \\ -\mu r \frac{V}{|V|}(p_{t_i} - rs_{t_i}) & 0 & -\mu \frac{V}{|V|}(p_{t_i} - rs_{t_i}) & 0 \end{bmatrix}$$

With previous classification, the function becomes differentiable except for those points where an equality condition is obtained, and this is a very unusual situation. The solution algorithm will be then organised following the next basic steps:

1. Initialise the iteration counter, $k = 0$, and start the iteration process with an initial and known solution $\mathbf{x}^0 = (\boldsymbol{\delta}_n^0, \mathbf{s}_t^0, \mathbf{p}_n^0, \mathbf{p}_t^0)$.

2. Solve the equation system $\boldsymbol{\Phi}(\mathbf{x}^k) + \partial\boldsymbol{\Phi}(\mathbf{x}^k)\mathbf{x}^{k+1} = 0$ to obtain $\mathbf{x}^{k+1}$.

3. If this is the final solution, $\left\| \boldsymbol{\Phi}(\mathbf{x}^{k+1}) \right\| < \varepsilon$, finish. In other case go to next point step.

4. Increment the iteration counter, $k = k+1$, and start again with step 2 using the solution computed in step 3 as initial solution.

6 Results

Three application examples in growing order of complexity are presented. First we treat the case of rolling between cylinders of the same material and radius (similar), in the second example the case of cylinders of different material (dissimilar) is exposed, and lastly the results for the case of rolling of a cylinder covered with a fine layer of elastic material (tyred), will be presented.

All the examples have been solved with models based in the half-space approach because obtained results will be compared with those reported by authors who have solved the same problems with other techniques, but using this approach.

Each cylinder has been discretized like a half-space with a potential contact zone of small dimensions compared with the characteristic dimensions of the domain, where the mesh has been refined to represent accurately the stress concentrations in its surroundings. In Fig. 2 the boundary element mesh used is represented in a schematic form, where the contact points of the two contact surfaces are in the same geometrical position.

Discontinuous quadratic elements have been used for the contact boundary mesh. Most of them are of the same size and they are evenly distributed by the different contact surfaces of the pattern. The other boundaries have been discretized with continuous elements, except at the corners where partial discontinuous elements are used (only at the extreme corresponding with the corner).

Because contact conditions are only verified at the element nodes, we have to accept the possibility of little incompatibilities derived from the change of contact status inside the element. This incompatibilities can be source of cumulative errors in path dependent contact problems, as reported in Man and Aliabadi (1993), but we have not found this problem in our results due to the path independent character of the steady-state regime.

Also the influence of different friction coefficients for the similar and dissimilar case is presented, proving the good behaviour of the discontinuous quadratic elements in this kind of problems.

6.1 Similar problems

The simplest rolling contact problem consists of two identical rotating cylinders. Obtained results will be compared with those computed using analytical solution from Carter (1926). This problem has also been solved by means of semi-analytic methods by Bentall and Johnson (1967) and Kalker (1971).

Modelled cylinders have a radius of $R = 100\,mm$, the friction coefficient is $\mu = 0.1$, the initial overlapping $\overline{\delta}_{AB} = 0.015\,mm.$, and the *creep* is such that $Q/\mu P = 0.51$, for the first case, and $Q/\mu P = 0.84$ for the second, being Q and P the total sum of forces, of the tangential and normal tractions acting on the contact area.

The boundary element mesh is shown in Fig. 2, where 21 discontinuous quadratic elements are used to represent the contact area displacements and tractions.
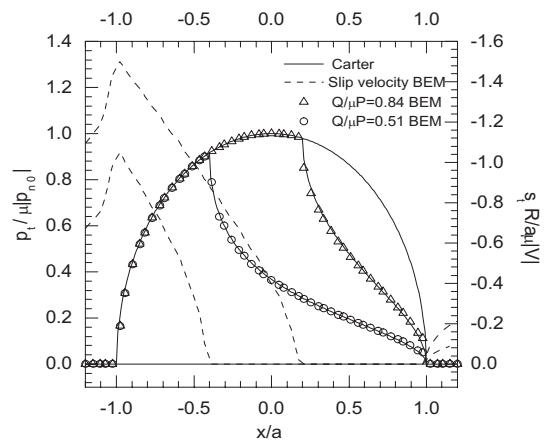

Figure 3 : Similar cylinders. Tangential tractions.

In Fig. 3 the tangential tractions obtained are compared with the solution of Carter and slip velocities obtained with the BEM are also represented. The normalisation factors are p_{n0}, maximum Hertz normal traction, and a, half length of the contact area.

Another case for similar cylinders can be found in Fig. 4, now a relation $Q/P = 0.051$ is maintained and tangential tractions for different friction coefficients ($\mu = 0.1, 0.2, 0.3$ and 0.4) are presented. As expected, the adhesion zone becomes larger when friction coefficient increases and only one slip-adhesion transition is obtained.
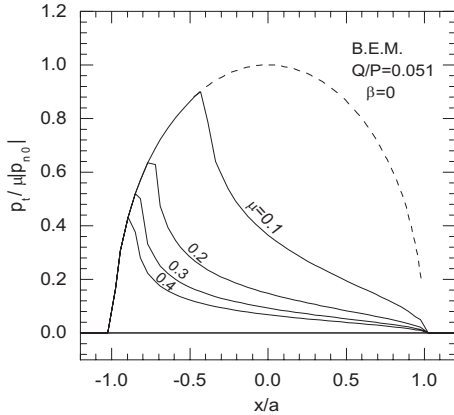
Figure 4 : Similar cylinders. Influence of friction coefficient on the tangential tractions.
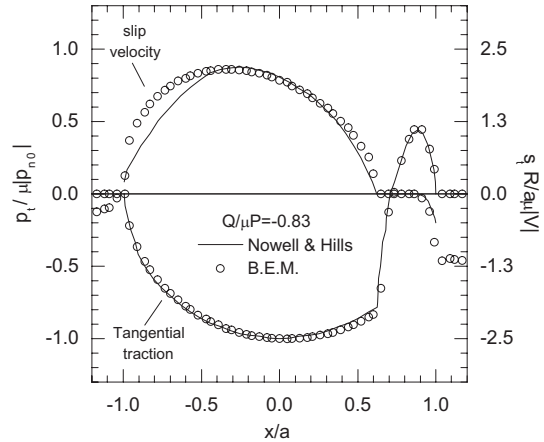

Figure 6 : Dissimilar cylinders. Two slip zones in opposite directions.

6.2 Dissimilar problems

When the cylinders are made of different materials, the problem becomes more complicated because several adhesion-slip transition points can appear. The Dundur's constant

$$\beta = \frac{\left[\left(1 - 2v^A\right)\left(1 + v^A\right)\right]\frac{E^B}{E^A} - \left[\left(1 - 2v^B\right)\left(1 + v^B\right)\right]}{\left(1 - v^{A2}\right)\frac{E^B}{E^A} + \left(1 - v^{B2}\right)} \quad (40)$$

quantifies similarity between both materials, and together with the friction coefficient and the tangential load, govern the distribution of the slip and adhesion zones.

For all problems parameters were: a friction coefficient $\mu = 0.1$, $\beta = 0.288$ (corresponding approximately to aluminium and steel), and an overlapping $\underline{\delta}_{AB} = 0.015\,mm$. The boundary element mesh coincides with the one used to solve the similar cylinders case. The tangential tractions and slip velocities obtained with the proposed formulation have been plotted in Figs. 5, 6, and 7, where they are compared with those obtained by Nowell and Hills (1987a) for values of $Q/\mu P$ equal to 0.75, -0.83 and 0.04.

For the case $Q/\mu P = 0.75$, Fig. 5, the agreement between tangential tractions is excellent, the situation of slip and adhesion zones is correct, however some discrepancies of the slip velocities are obtained.
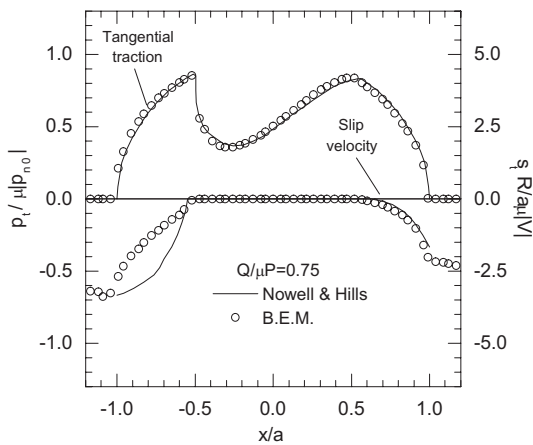

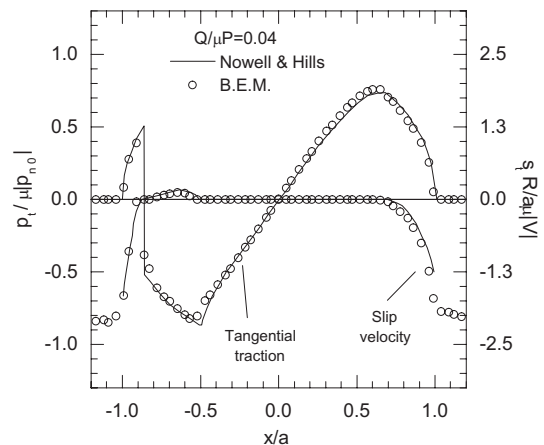**Figure 5** : Dissimilar cylinders. Two slip zones in equal direction.

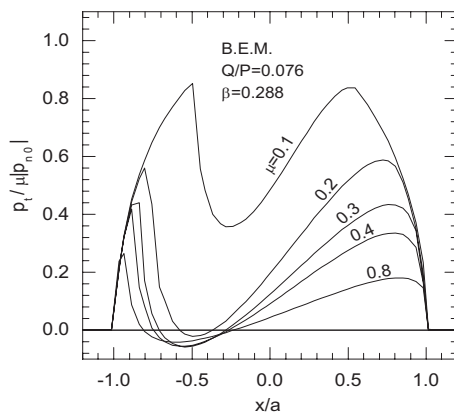**Figure 7** : Dissimilar cylinders. Three slip zones.

Figure 8 : Dissimilar cylinders. Influence of friction coefficient on the tangential tractions.

In order to prove the good behaviour of the quadratic elements also in complex traction distributions, a dissimilar problem with a constant relation $Q/P = 0.076$ and multiple friction coefficients ($\mu = 0.1, 0.2, 0.3, 0.4$ and 0.8), is solved using the same mesh (Fig. 8). It can be observed that strong tangential traction variations can be well represented using discontinuous quadratic elements for this type of problem, if enough number of elements are used.

6.3   Tyred cylinder

In this last example we suppose that one of the cylinders used in the previous problems is covered with a fine layer of a much less rigid elastic material than the material of the cylinders. The layer thickness is called b, similar to the size of the contact area, but much smaller than the radius of the cylinders.

This problem was solved by Nowell and Hills (1987b) using semi-analytic methods, based on a previous work of Bentall and Johnson (1968). To be able to compare our results with other authors, we will assume their hypotheses assuming that the two cylinders are rigid bodies, and one of them with radius R rotates on a layer of elastic material attached peripherally to the other, Fig. 9.

From the BEM standpoint, it only necessary to discretise the layer boundary using large enough number of elements (Fig. 10) and to assume that the normal load is applied by means of an overlap $\overline{\delta}_{AB}$ between the cylinders.

The results for this problem are presented in Fig. 11. They correspond to a friction coefficient $\mu = 0.1$, a relationship $b/a = 0.1$, the Poisson's ratio is $\nu = 0.5$ (incompressible material), and relations between normal and tangential loads, $Q/\mu P$, of $-0.74, 0.41$ and 0.05. The curves presented have been centred to facilitate their comparison, since for the present problem the centre of the contact area suffers a shift due to the tangential
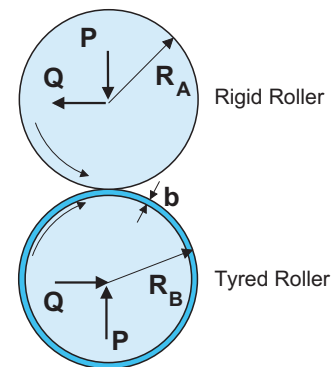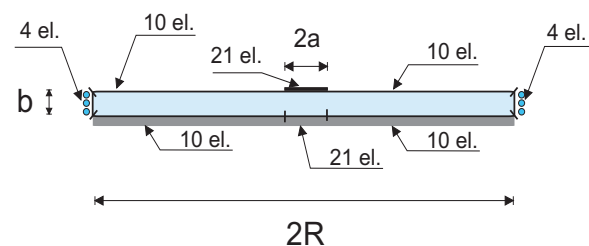

Figure 9 : Tyred cylinder.

Figure 10 : Model and discretization used for the tyred cylinder.

load. Nowell and Hills carried out a similar process with their results.

7   Conclusions

A new methodology for the solution of the steady-state two-dimensional rolling problem between cylinders, based on Mathematical Programming Techniques and on the Boundary Element Method, has been proposed.

In this methodology, the influence coefficients of the elastic equations have been obtained by means of BEM, and the highly non-linear problem associated with rolling contact has been solved using a formulation inspired by the Augmented Lagrangian formulation of the contact problem.

Convergence of the Mathematical Programming problem has been improved in comparison with other algorithms experienced by the authors, based on complementarity relations or on quadratic mathematical programming methods.

The algorithm has been shown to be very efficient, it simplifies the procedure to obtain the solution of a rolling problem, and minimises the physical knowledge of the particular problem needed to control the convergence of the non-linear iteration process, and is an effective alternative to the actual algorithms.

Definitely, the algorithm presented in this paper simplifies the solution of rolling contact problems, more if we think about
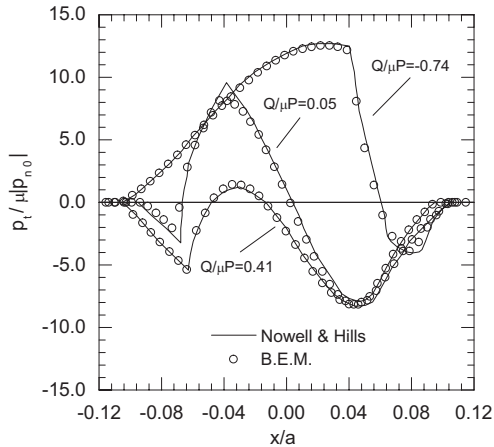
Figure 11 : Tangential tractions on the tyre.

the necessary effort to solve the same problem by means of trial and error or semi-analytic methods.

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