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Flexural - Torsional Nonlinear Analysis of Timoshenko Beam-Column of Arbitrary Cross Section by BEM

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Abstract: In this paper a boundary element method is developed for the nonlinear flexural – torsional analysis of Timoshenko beam-columns of arbitrary simply or multiply connected constant cross section, undergoing moderate large deflections under general boundary conditions. The beam-column is subjected to the combined action of an arbitrarily distributed or concentrated axial and transverse loading as well as to bending and twisting moments. To account for shear deformations, the concept of shear deformation coefficients is used. Seven boundary value problems are formulated with respect to the transverse displacements, to the axial displacement, to the angle of twist (which is assumed to be small), to the primary warping function and to two stress functions and solved using the Analog Equation Method, a BEM based method. Application of the boundary element technique yields a system of nonlinear equations from which the transverse and axial displacements as well as the angle of twist are computed by an iterative process. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. Numerical examples with great practical interest are worked out to illustrate the efficiency, the accuracy and the range of applications of the developed method. The influence of both the shear deformation effect and the variableness of the axial loading are remarkable.

Keywords: Flexural-Torsional Analysis; Timoshenko beam-column; Shear center; Shear deformation coefficients; Nonlinear analysis; Boundary element method

1 Introduction

In recent years, a need has been raised in the analysis of the components of plane and space frames or grid systems to take into account the influence of the action of axial, lateral forces and end moments on their deformed shape. Lateral loads and end moments generate deflection that is further amplified by axial compression

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loading. The aforementioned analysis becomes much more accurate and complex taking into account that the axial force is nonlinearly coupled with the transverse deflections. This non-linearity results from retaining the square of the slope in the strain–displacement relations (intermediate non-linear theory), avoiding in this way the inaccuracies arising from a linearized second – order analysis. Moreover, unless the beam is very "thin" the error incurred from the ignorance of the effect of shear deformation may be substantial, particularly in the case of heavy lateral loading.

Over the past thirty years, many researchers have developed and validated various methods of performing a nonlinear analysis on structures with pioneer the work of Gobarah and Tso who developed a theoretical model based on shell theory, leading to highly coupled nonlinear equations for thin-walled elements [Gobarah and Tso (1971)] and Black who used non-linear torque-rotation relations in his analysis [Black (1967)]. Consequently, all of the research efforts employed the finite element method for the nonlinear beam analysis. More specifically, Mondkar and Powell applying the Principle of Virtual Displacements [Mondkar and Powell (1977)] developed a general formulation of the incremental equations of motion for structures undergoing large displacement, finite strain deformation, Argyris et al. introduced the notion of semitangential rotation to avoid the difficulty arising from the noncommutative nature of rotations about fixed axes with different directions [Argyris, Dunne, Malejannakis and Scharpf (1978a, 1978b, 1979)], Bathe and Bolourchi presented an updated Lagrangian and a total Lagrangian formulation of a three-dimensional beam element for large displacement and large rotation analysis [Bathe and Bolourchi (1979)], Attard in [Attard (1986)] and Attard and Somervaille in [Attard and Somervaille (1987)] have developed a set of displacement relationships for a straight prismatic thin-walled open beam applicable to situations where displacements are finite, the cross section does not distort, strains are small and flexural displacements are small to moderate while cross-sectional twist can be large, presenting examples of either mono-symmetric or doubly-symmetric cross sections, Yang and McGuire performed a nonlinear analysis using an updated Lagrangian finite element derivation for doubly-symmetric sections [Yang and McGuire (1986)], starting from the principle of virtual displacements, Ronagh et. al. developed a theory for thin-walled elements with variable cross sections based on a nonlinear formulation for the Green-Lagrange axial strain [Ronagh, Bradford and Attard (2000a, 2000b)], Conci formulated a stiffness matrix using the updated Lagrangian procedure for space frames with generic thin-walled cross sections [Conci (1992)], Mohri et. al. developed a large torsion finite element model for thin-walled Bernoulli beams [Mohri, Azrar and Potier-Ferry (2008)] while recently Cai et al. presented a simple finite element method, based on a von Karman type nonlinear theory of deformation, for geometrically nonlinear large rotation analyses of space frames consisting of members of arbitrary cross-section [Cai, Paik and Atluri (2009)]. Nevertheless, in all of the aforementioned research efforts the analysis is not general since either the axial loading of the structural components is assumed to be constant, or the analysis is restricted to the thin-walled theory assumptions or to mono-symmetric or doubly-symmetric cross sections, while shear deformation effect has been ignored.

Shear deformation has been extensively examined in either static or dynamic linear analysis of beams [Nadolski and Pielorz (1992); Marchionna and Panizzi (1997); Raffa and Vatta (2001); Ozdemir Ozgumus and Kaya (2010)]. Concerning nonlinear analysis, Reissner studied the one dimensional equilibrium problem (plane problem) and the lateral buckling of beams including shear deformation [Reissner (1972, 1979)], while in the work of Omar and Shabana [Omar and Shabana (2001)], Dufva et. al [Dufva, Sopanen and Mikkola (2005)] and Aristizabal-Ochoa [Aristizabal-Ochoa (2008)] the aforementioned effect is taken into account in their analysis which is however restricted to symmetrical cross sections (no torsion is considered).

Finally, the boundary element method has also been employed for the nonlinear analysis of beams. A BEM-based method has been presented by Katsikadelis and Tsiatas [Katsikadelis and Tsiatas (2003); Katsikadelis and Tsiatas (2004)] for the nonlinear static and dynamic flexural analysis of beams of symmetrical cross section with variable stiffness ignoring shear deformation effect and by Sapountzakis and Mokos [Sapountzakis and Mokos (2008)], Sapountzakis and Panagos [Sapountzakis and Panagos (2008)] for the nonlinear flexural analysis of Timoshenko beams of doubly-symmetric constant or variable cross section (in all of these research efforts no torsion is considered). The boundary element method has not yet been used for the nonlinear flexural – torsional analysis of beams taking into account shear deformation effect.

In this paper, which is an extension of the aforementioned work of the first author [Sapountzakis and Mokos (2008); Sapountzakis and Panagos (2008)], a boundary element method is developed for the nonlinear flexural – torsional analysis of Timoshenko beam-columns of arbitrary simply or multiply connected constant cross section, undergoing moderate large deflections under general boundary conditions. The beam-column is subjected to the combined action of an arbitrarily distributed or concentrated axial and transverse loading as well as to bending and twisting moments. To account for shear deformations, the concept of shear deformation coefficients is used. Seven boundary value problems are formulated with respect to the transverse displacements, to the axial displacement, to the angle of twist (which is assumed to be small), to the primary warping function and to two stress functions and solved using the Analog Equation Method [Katsikadelis (2002)], a BEM based method. Application of the boundary element technique yields a system of nonlinear equations from which the transverse and axial displacements as well as the angle of twist are computed by an iterative process. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. It should be mentioned that due to the assumption of small angles of twist, out of plane instabilities cannot be examined by the proposed method. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

- The beam-column is subjected to an arbitrarily distributed or concentrated axial and transverse loading as well as to bending and twisting moments.
- The beam-column is supported by the most general boundary conditions including elastic support or restraint, while its cross section is an arbitrary one.
- The proposed formulation is applicable to arbitrarily shaped thin or thick-walled cross sections occupying simple or multiple connected domains, taking into account shear deformation effect. The employed method is based on the Timoshenko approach, which is presented as compared with the Engesser or Haringx methods in [Atanackovic and Spasic (1994)].
- The present formulation does not stand on the assumption of a thin-walled structure and therefore the cross section's torsional rigidity is evaluated exactly without using the so-called Saint –Venant's torsional constant.
- The shear deformation coefficients are evaluated using an energy approach, instead of Timoshenko's [Timoshenko (1921)] and Cowper's definitions [Cowper (1966)], for which several authors [Schramm, Kitis, Kang and Pilkey (1994); Schramm, Rubenchik and Pilkey (1997)] have pointed out that one obtains unsatisfactory results or definitions given by other researchers [Stephen (1980); Hutchinson (2001)], for which these factors take negative values.
- The effect of the material's Poisson ratio ν is taken into account.

The proposed method employs a pure BEM approach (requiring only boundary discretization) resulting in line or parabolic elements instead of area elements of the FEM solutions (requiring the whole cross section to be discretized into triangular or quadrilateral area elements), while a small number of line elements are required to achieve high accuracy.

Numerical examples are worked out to illustrate the efficiency, the accuracy and the range of applications of the developed method. The influence of the shear deformation effect is observed. The obtained numerical results are compared with those obtained from a 3-D FEM solution using solid elements.

2 Statement of the problem

Let us consider a prismatic beam-column of length l (Fig.1), of constant arbitrary cross-section of area A. The homogeneous isotropic and linearly elastic material of the beam cross-section, with modulus of elasticity$E$, shear modulus G and Poisson's ratio v occupies the two dimensional multiply connected region Ω of the y, z plane and is bounded by the Γ_j $(j = 1, 2, ..., K)$ boundary curves, which are piecewise smooth, i.e. they may have a finite number of corners. In Fig. 1a CYZ is the principal shear system of axes through the cross section's centroid C, while y_C, z_C are its coordinates with respect to the Syz system through the cross section's shear center S, with axes parallel to those of CYZ. The beam-column is subjected to the combined action of the arbitrarily distributed or concentrated axial loading $p_X = p_X(X)$, transverse loading $p_Y = p_Y(X)$, $p_Z = p_Z(X)$ acting in the Y and Z directions, respectively, bending moments $m_Y = m_Y(X)$, $m_Z = m_Z(X)$ along Y and Z axes, respectively and twisting moment $m_x = m_x(x)$ (Fig. 1b).

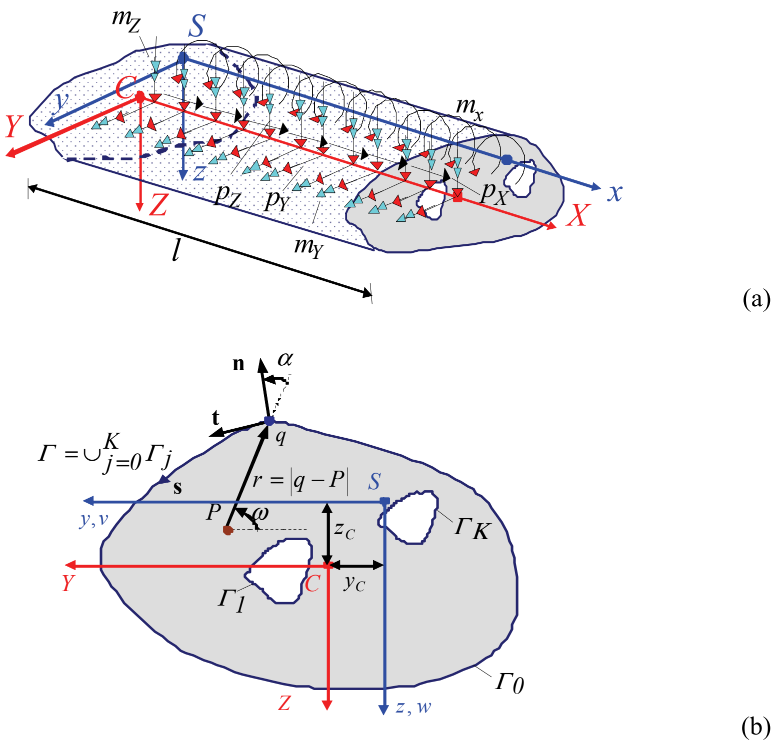

Figure 1: Prismatic beam-column in axial - flexural – torsional loading (a) with an arbitrary cross-section occupying the two dimensional region Ω (b).

Under the action of the aforementioned loading, the displacement field of the beam-column, assuming small angle of twist θ_x, that is $\cos\theta_x \approx 1$, $\sin\theta_x \approx \theta_x$ and ignoring the resulting nonlinear terms of the small angle of twist, is given as [Sapountzakis and Dourakopoulos (2009)]

$$\bar{u}(x,y,z) = u(x) + \theta_Y(x) Z - \theta_Z(x) Y + \theta_x'(x)\,\phi_S^P(y,z) + \phi_S^S(x,y,z) \tag{1a}$$

$$\bar{v}(x,z) = v(x) - z\theta_x(x) \tag{1b}$$

$$\bar{w}(x,y) = w(x) + y\theta_x(x) \tag{1c}$$

where $\bar{u}$, $\bar{v}$, $\bar{w}$ are the axial and transverse beam displacement components with respect to the Syz shear system of axes; $v = v(x)$, $w = w(x)$ are the corresponding components of the shear center S; $u = u(x)$ denotes the average longitudinal displacement of the cross section [Attard (1986)]; θ_Y, θ_Z are the angles of rotation due to bending of the cross-section; $\theta_x'(x)$ denotes the rate of change of the angle of twist θ_x regarded as the torsional curvature and ϕ_S^P, φ_S^S are the primary and secondary warping functions with respect to the shear center S [Sapountzakis and Mokos (2003)].

Employing the strain-displacement relations of the three - dimensional elasticity for moderate displacements [Ramm and Hofmann (1995), Rothert and Gensichen (1987)], the following strain components can be easily obtained

$$\varepsilon_{xx} = \frac{\partial \bar{u}}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial \bar{v}}{\partial x}\right)^2 + \left(\frac{\partial \bar{w}}{\partial x}\right)^2\right] \tag{2a}$$

$$\gamma_{xz} = \frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{u}}{\partial z} \tag{2b}$$

$$\gamma_{xy} = \frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y} \tag{2c}$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \gamma_{yz} = 0 \tag{2d}$$

Following the procedure presented in [Sapountzakis and Dourakopoulos (2008)] the three coupled differential equations of equilibrium of the beam-column under consideration, subjected to the combined action of axial, bending and torsional loading are obtained as

$$
\begin{aligned}
&EI_{ZZ}\frac{d^4v}{dx^4}+EI_{YZ}\frac{d^4w}{dx^4}-p_Y+p_X\left(\frac{dv}{dx}-z_C\frac{d\theta_x}{dx}\right)-N\left(\frac{d^2v}{dx^2}-z_C\frac{d^2\theta_x}{dx^2}\right)+\frac{dm_Z}{dx}+\\
&+\frac{EI_{ZZ}}{GA_Y}\left[\frac{d^2p_Y}{dx^2}-\frac{d^2p_X}{dx^2}\left(\frac{dv}{dx}-z_C\frac{d\theta_x}{dx}\right)-3\frac{dp_X}{dx}\left(\frac{d^2v}{dx^2}-z_C\frac{d^2\theta_x}{dx^2}\right)-\right.\\
&\left.-3p_X\left(\frac{d^3v}{dx^3}-z_C\frac{d^3\theta_x}{dx^3}\right)+N\left(\frac{d^4v}{dx^4}-z_C\frac{d^4\theta_x}{dx^4}\right)\right]+\\
&+\frac{EI_{YZ}}{GA_Z}\left[\frac{d^2p_Z}{dx^2}-\frac{d^2p_X}{dx^2}\left(\frac{dw}{dx}+y_C\frac{d\theta_x}{dx}\right)-3\frac{dp_X}{dx}\left(\frac{d^2w}{dx^2}+y_C\frac{d^2\theta_x}{dx^2}\right)-\right.\\
&\qquad\qquad\left.-3p_X\left(\frac{d^3w}{dx^3}+y_C\frac{d^3\theta_x}{dx^3}\right)+N\left(\frac{d^4w}{dx^4}+y_C\frac{d^4\theta_x}{dx^4}\right)\right]=0
\end{aligned}
\tag{3a}
$$

$$
\begin{aligned}
&EI_{YY}\frac{d^4w}{dx^4}+EI_{YZ}\frac{d^4v}{dx^4}-p_Z+p_X\left(\frac{dw}{dx}+y_C\frac{d\theta_x}{dx}\right)-N\left(\frac{d^2w}{dx^2}+y_C\frac{d^2\theta_x}{dx^2}\right)-\frac{dm_Y}{dx}+\\
&+\frac{EI_{YY}}{GA_Z}\left[\frac{d^2p_Z}{dx^2}-\frac{d^2p_X}{dx^2}\left(\frac{dw}{dx}+y_C\frac{d\theta_x}{dx}\right)-3\frac{dp_X}{dx}\left(\frac{d^2w}{dx^2}+y_C\frac{d^2\theta_x}{dx^2}\right)-\right.\\
&\left.-3p_X\left(\frac{d^3w}{dx^3}+y_C\frac{d^3\theta_x}{dx^3}\right)+N\left(\frac{d^4w}{dx^4}+y_C\frac{d^4\theta_x}{dx^4}\right)\right]+\\
&+\frac{EI_{YZ}}{GA_Y}\left[\frac{d^2p_Y}{dx^2}-\frac{d^2p_X}{dx^2}\left(\frac{dv}{dx}-z_C\frac{d\theta_x}{dx}\right)-3\frac{dp_X}{dx}\left(\frac{d^2v}{dx^2}-z_C\frac{d^2\theta_x}{dx^2}\right)-\right.\\
&\qquad\qquad\left.-3p_X\left(\frac{d^3v}{dx^3}-z_C\frac{d^3\theta_x}{dx^3}\right)+N\left(\frac{d^4v}{dx^4}-z_C\frac{d^4\theta_x}{dx^4}\right)\right]=0
\end{aligned}
\tag{3b}
$$

$$
\begin{aligned}
&EC_S\frac{d^4\theta_x}{dx^4}-GI_t\frac{d^2\theta_x}{dx^2}-N\left(y_C\frac{d^2w}{dx^2}-z_C\frac{d^2v}{dx^2}+\frac{\mathrm{I}_S}{A}\frac{d^2\theta_x}{dx^2}\right)=\\
&\qquad\qquad m_x+p_Zy_C-p_Yz_C-p_X\left(y_C\frac{dw}{dx}-z_C\frac{dv}{dx}\right)-p_X\frac{\mathrm{I}_S}{A}\frac{d\theta_x}{dx}
\end{aligned}
\tag{3c}
$$

where

$$
A_Z=\kappa_ZA=\frac{1}{a_Z}A \quad A_Y=\kappa_YA=\frac{1}{a_Y}A \tag{4}
$$

are the shear areas with respect to Y, Z axes, respectively with κ_Y, κ_z the shear correction factors, a_Y, a_Z the shear deformation coefficients and A the cross section area. Moreover, I_S is the polar moment of inertia with respect to the shear center S, EC_S and GI_t are the cross section's warping and torsional rigidities, respectively, with C_S, I_t being its warping and torsion constants, respectively, given as [Sapountzakis and Mokos (2003)]

$$C_S = \int_\Omega \left(\varphi_S^P\right)^2 d\Omega \tag{5}$$

$$I_t = \int_\Omega \left(y^2 + z^2 + y\frac{\partial \varphi_S^P}{\partial z} - z\frac{\partial \varphi_S^P}{\partial y}\right) d\Omega \tag{6}$$

It is worth here noting that the primary warping function $\varphi_S^P(y,z)$ can be established by solving independently the Neumann problem [Sapountzakis and Mokos (2003)]

$$\nabla^2 \varphi_S^P = 0 \text{ in } \Omega \tag{7}$$

$$\frac{\partial \varphi_S^P}{\partial n} = \frac{1}{2}\frac{\partial\left(r_S^2\right)}{\partial s} \text{ on } \Gamma_j \ (j = 1,2,...,K) \tag{8}$$

where $\nabla^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplace operator; $r_S = \sqrt{y^2 + z^2}$ is the distance of a point on the boundary Γ_j from the shear center S; $\partial/\partial n$ denotes the directional derivative normal to the boundary Γ_j and $\partial/\partial s$ denotes differentiation with respect to its arc length s.

The aforementioned governing differential equations are also subjected to the pertinent boundary conditions of the problem, which are given as

$$\alpha_1 v(x) + \alpha_2 V_y(x) = \alpha_3 \quad \bar{\alpha}_1 \theta_Z(x) + \bar{\alpha}_2 \mathrm{M}_Z(x) = \bar{\alpha}_3 \tag{9}$$

$$\beta_1 w(x) + \beta_2 V_z(x) = \beta_3 \quad \bar{\beta}_1 \theta_Y(x) + \bar{\beta}_2 \mathrm{M}_Y(x) = \bar{\beta}_3 \tag{10}$$

$$\gamma_1 \theta_x(x) + \gamma_2 M_t(x) = \gamma_3 \quad \bar{\gamma}_1 \frac{d\theta_x(x)}{dx} + \bar{\gamma}_2 M_w(x) = \bar{\gamma}_3 \tag{11}$$

at the beam ends $x = 0, l$, where V_y, V_z and M_Z, M_Y are the reactions and bending moments with respect to y, z and Y, Z axes, respectively, given as

$$\begin{aligned} V_y = & -EI_{ZZ}\frac{d^3v}{dx^3} - \frac{EI_{ZZ}}{GA_Y}N\left(\frac{d^3v}{dx^3} - z_C\frac{d^3\theta_x}{dx^3}\right) - EI_{YZ}\frac{d^3w}{dx^3} - \\ & \frac{EI_{YZ}}{GA_Z}N\left(\frac{d^3w}{dx^3} + y_C\frac{d^3\theta_x}{dx^3}\right) + N\left(\frac{dv}{dx} - z_C\frac{d\theta_x}{dx}\right) \end{aligned} \tag{12}$$

$$\begin{aligned} V_z = & -EI_{YY}\frac{d^3w}{dx^3} - \frac{EI_{YY}}{GA_Z}N\left(\frac{d^3w}{dx^3} + y_C\frac{d^3\theta_x}{dx^3}\right) - EI_{YZ}\frac{d^3v}{dx^3} - \\ & \frac{EI_{YZ}}{GA_Y}N\left(\frac{d^3v}{dx^3} - z_C\frac{d^3\theta_x}{dx^3}\right) + N\left(\frac{dw}{dx} + y_C\frac{d\theta_x}{dx}\right) \end{aligned} \tag{13}$$

$$
\begin{aligned}
M_Y = & -EI_{YY}\frac{d^2w}{dx^2}-\frac{EI_{YY}}{GA_Z}N\left(\frac{d^2w}{dx^2}+y_C\frac{d^2\theta_x}{dx^2}\right)\\
& -EI_{YZ}\frac{d^2v}{dx^2}-\frac{EI_{YZ}}{GA_Y}N\left(\frac{d^2v}{dx^2}-z_C\frac{d^2\theta_x}{dx^2}\right)
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
M_Z = & EI_{ZZ}\frac{d^2v}{dx^2}+\frac{EI_{ZZ}}{GA_Y}N\left(\frac{d^2v}{dx^2}-z_C\frac{d^2\theta_x}{dx^2}\right)\\
& +EI_{YZ}\frac{d^2w}{dx^2}+\frac{EI_{YZ}}{GA_Z}N\left(\frac{d^2w}{dx^2}+y_C\frac{d^2\theta_x}{dx^2}\right)
\end{aligned}
\tag{15}
$$

θ_Y, θ_Z are the angles of rotation due to bending given as

$$
\begin{aligned}
\theta_Y = & -\frac{dw}{dx}-\frac{EI_{YY}}{GA_Z}\frac{d^3w}{dx^3}-\frac{EI_{YY}}{G^2A_Z^2}N\left(\frac{d^3w}{dx^3}+y_C\frac{d^3\theta_x}{dx^3}\right)-\frac{EI_{YZ}}{GA_Z}\frac{d^3v}{dx^3}-\\
& -\frac{EI_{YZ}}{G^2A_YA_Z}N\left(\frac{d^3v}{dx^3}-z_C\frac{d^3\theta_x}{dx^3}\right)
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\theta_Z = & \frac{dv}{dx}+\frac{EI_{ZZ}}{GA_Y}\frac{d^3v}{dx^3}+\frac{EI_{ZZ}}{G^2A_Y^2}N\left(\frac{d^3v}{dx^3}-z_C\frac{d^3\theta_x}{dx^3}\right)+\frac{EI_{YZ}}{GA_Y}\frac{d^3w}{dx^3}+\\
& +\frac{EI_{YZ}}{G^2A_YA_Z}N\left(\frac{d^3w}{dx^3}+y_C\frac{d^3\theta_x}{dx^3}\right)
\end{aligned}
\tag{17}
$$

while in eqns. (11) M_t and M_w are the torsional and warping moments at the boundary of the bar, respectively, given as

$$
M_t = -EC_S\frac{d^3\theta_x}{dx^3}+GI_t\frac{d\theta_x}{dx}+N\left(y_C\frac{dw}{dx}-z_C\frac{dv}{dx}+\frac{I_S}{A}\frac{d\theta_x}{dx}\right)
\tag{18}
$$

$$
M_w = -EC_S\frac{d^2\theta_x}{dx^2}
\tag{19}
$$

Finally, $\alpha_k, \bar{\alpha}_k, \beta_k, \bar{\beta}_k, \gamma_k, \bar{\gamma}_k$ $(k=1,2,3)$ are functions specified at the beam ends $x=0,l$. Eqs. (9)-(11) describe the most general boundary conditions associated with the problem at hand and can include elastic support or restraint. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived from these equations by specifying appropriately these functions (e.g. for a clamped edge it is $\alpha_1=\beta_1=\gamma_1=1$, $\overline{\alpha}_1=\overline{\beta}_1=\overline{\gamma}_1=1$, $\alpha_2=\alpha_3=\beta_2=\beta_3=\gamma_2=\gamma_3=\overline{\alpha}_2=\overline{\alpha}_3=\overline{\beta}_2=\overline{\beta}_3=\overline{\gamma}_2=\overline{\gamma}_3=0$).

In the aforementioned boundary value problem the axial force N inside the beam or at its boundary, neglecting normal deformation due to torsion, is given from the following relation

$$N = EA\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2\right] \tag{20}$$

where $u = u(x)$ is the bar axial displacement, which can be evaluated from the solution of the following boundary value problem

$$EA\left[\frac{d^2u}{dx^2} + \frac{d^2w}{dx^2}\frac{dw}{dx} + \frac{d^2v}{dx^2}\frac{dv}{dx}\right] = -p_x \text{ inside the beam} \tag{21}$$

$$c_1 u(x) + c_2 N(x) = c_3 \text{ at the beam ends } x = 0, 1 \tag{22}$$

where c_i $(i = 1, 2, 3)$ are given constants.

The solution of the boundary value problem given from eqns (3), (21) subjected to the boundary conditions (9)-(11), (22) which represents the nonlinear flexural – torsional analysis of beam-columns, presumes the evaluation of the shear deformation coefficients a_Y, a_Z, corresponding to the principal shear axes coordinate system. These coefficients are established equating the approximate formula of the shear strain energy per unit length [Schramm, Rubenchik and Pilkey (1997)]

$$U_{appr.} = \frac{a_Y Q_y^2}{2AG} + \frac{a_Z Q_z^2}{2AG} \tag{23}$$

with the exact one given from

$$U_{exact} = \int_\Omega \frac{(\tau_{xz})^2 + (\tau_{xy})^2}{2G} d\Omega \tag{24}$$

and are obtained as [Sapountzakis and Mokos (2005)]

$$a_Y = \frac{1}{\kappa_Y} = \frac{A}{\Delta^2}\int_\Omega [(\nabla\Theta) - e] \cdot [(\nabla\Theta) - e]\, d\Omega \tag{25a}$$

$$a_Z = \frac{1}{\kappa_Z} = \frac{A}{\Delta^2}\int_\Omega [(\nabla\Phi) - d] \cdot [(\nabla\Phi) - d]\, d\Omega \tag{25b}$$

where $(\tau_{xz})_j, (\tau_{xy})_j$ are the transverse (direct) shear stress components, $(\nabla) \equiv i_Y(\partial/\partial Y) + i_Z(\partial/\partial Z)$ is a symbolic vector with i_Y, i_Z the unit vectors along Y and Z axes, respectively, Δ is given from

$$\Delta = 2(1+\nu)\left(\mathrm{I}_{YY}\mathrm{I}_{ZZ} - I_{YZ}^2\right) \tag{26}$$

ν is the Poisson ratio of the cross section material, e and d are vectors defined as

$$e = \left[\nu\left(I_{YY}\frac{Y^2-Z^2}{2} - I_{YZ}YZ\right)\right] i_Y + \left[\nu\left(I_{YY}YZ + I_{YZ}\frac{Y^2-Z^2}{2}\right)\right] i_Z \tag{27}$$

$$d = \left[\nu\left(I_{ZZ}YZ - I_{YZ}\frac{Y^2-Z^2}{2}\right)\right] i_Y + \left[-\nu\left(I_{ZZ}\frac{Y^2-Z^2}{2} + I_{YZ}YZ\right)\right] i_Z \tag{28}$$

and $\Theta(Y,Z)$, $\Phi(Y,Z)$ are stress functions, which are evaluated from the solution of the following Neumann type boundary value problems [Sapountzakis and Mokos (2005)]

$$\nabla^2\Theta = 2\left(I_{YZ}Z - I_{YY}Y\right) \text{ in } \Omega \tag{29a}$$

$$\frac{\partial\Theta}{\partial n} = n\cdot e \text{ on } \Gamma = \bigcup_{j=1}^{K+1}\Gamma_j \tag{29b}$$

$$\nabla^2\Phi = 2\left(I_{YZ}Y - I_{ZZ}Z\right) \text{ in } \Omega \tag{30a}$$

$$\frac{\partial\Phi}{\partial n} = n\cdot d \text{ on } \Gamma = \bigcup_{j=1}^{K+1}\Gamma_j \tag{30b}$$

where n is the outward normal vector to the boundary Γ. In the case of negligible shear deformations $a_Z = a_Y = 0$. It is also worth here noting that the boundary conditions (8), (29b), (30b) have been derived from the physical consideration that the traction vector in the direction of the normal vector n vanishes on the free surface of the beam.

3 Integral Representations - Numerical Solution

According to the precedent analysis, the nonlinear flexural – torsional analysis of Timoshenko beam-columns of arbitrary cross section, undergoing moderate large deflections reduces in establishing the displacement components $v(x)$, $w(x)$ and $\theta_x(x)$ having continuous derivatives up to the fourth order with respect to x and the axial displacement $u = u(x)$ having continuous derivatives up to the second order with respect to x satisfying the coupled governing equations (3), (21) inside the beam and the boundary conditions (9)-(11), (22) at the beam ends $x = 0, l$.

3.1 For the transverse displacements w, v and the angle of twist $\boldsymbol{\theta}_\mathbf{x}$

Eqns (3), (21) are solved using the Analog Equation Method [Katsikadelis (2002)]. This method has been developed for the beam equation including axial forces by

Katsikadelis and Tsiatas [Katsikadelis and Tsiatas (2004)]. However, the formulation presented in Sapountzakis and Mokos [Sapountzakis and Mokos (2008)] and in Sapountzakis and Panagos [Sapountzakis and Panagos (2008)] is followed in this investigation.

Let $v(x)$, $w(x)$ and $\theta_x(x)$ be the sought solution of the aforementioned boundary value problem. Setting as $u_2(x) = v(x)$, $u_3(x) = w(x)$, $u_4(x) = \theta_x(x)$ and differentiating these functions four times with respect to x yields

$$\frac{d^4 u_i}{dx^4} = q_i(x) \quad (i = 2, 3, 4) \tag{31}$$

Eqns. (31) indicate that the solution of eqns. (3) can be established by solving eqns. (31) under the same boundary conditions (9)-(11), provided that the fictitious load distributions $q_i(x)$ $(i = 2, 3, 4)$ are first established. These distributions can be determined using BEM.

Following the procedure presented in [Sapountzakis and Mokos (2008), Sapountzakis and Panagos (2008)] and employing the constant element assumption for the load distributions q_i along the L internal beam elements (as the numerical implementation becomes very simple and the obtained results are of high accuracy), the integral representations of the displacement components u_i $(i = 2, 3, 4)$ and their first derivatives with respect to x when applied for the beam ends $(0, l)$, together with the boundary conditions (9-11) are employed to express the unknown coupled boundary quantities $u_i(\zeta)$, $u_{i,x}(\zeta)$, $u_{i,xx}(\zeta)$ and $u_{i,xxx}(\zeta)$ $(\zeta = 0, l)$ in terms of q_i as

$$\begin{bmatrix}
\mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{0} & \mathbf{D}_{14} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{18} & 0 & \mathbf{D}_{1\,10} & 0 & \mathbf{D}_{1\,12} \\
0 & \mathbf{D}_{22} & \mathbf{D}_{23} & \mathbf{D}_{24} & 0 & 0 & \mathbf{D}_{27} & \mathbf{D}_{28} & 0 & 0 & \mathbf{D}_{2\,11} & \mathbf{D}_{2\,12} \\
\mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} & \mathbf{E}_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{E}_{42} & \mathbf{E}_{43} & \mathbf{E}_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{D}_{54} & \mathbf{D}_{55} & \mathbf{D}_{56} & 0 & \mathbf{D}_{58} & 0 & \mathbf{D}_{5\,10} & 0 & \mathbf{D}_{5\,12} \\
0 & 0 & \mathbf{D}_{63} & \mathbf{D}_{64} & 0 & \mathbf{D}_{66} & \mathbf{D}_{67} & \mathbf{D}_{68} & 0 & 0 & \mathbf{D}_{6\,11} & \mathbf{D}_{6\,12} \\
0 & 0 & 0 & 0 & \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} & \mathbf{E}_{34} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{E}_{42} & \mathbf{E}_{43} & \mathbf{E}_{44} & 0 & 0 & 0 & 0 \\
0 & \mathbf{D}_{92} & 0 & 0 & 0 & \mathbf{D}_{96} & 0 & 0 & \mathbf{D}_{99} & \mathbf{D}_{9\,10} & 0 & \mathbf{D}_{9\,12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{D}_{10\,10} & \mathbf{D}_{10\,11} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} & \mathbf{E}_{34} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{E}_{42} & \mathbf{E}_{43} & \mathbf{E}_{44}
\end{bmatrix}$$

$$\left\{\begin{matrix} \hat{\mathbf{u}}_2 \\ \hat{\mathbf{u}}_{2,x} \\ \hat{\mathbf{u}}_{2,xx} \\ \hat{\mathbf{u}}_{2,xxx} \\ \hat{\mathbf{u}}_3 \\ \hat{\mathbf{u}}_{3,x} \\ \hat{\mathbf{u}}_{3,xx} \\ \hat{\mathbf{u}}_{3,xxx} \\ \hat{\mathbf{u}}_4 \\ \hat{\mathbf{u}}_{4,x} \\ \hat{\mathbf{u}}_{4,xx} \\ \hat{\mathbf{u}}_{4,xxx} \end{matrix}\right\} = \left\{\begin{matrix} \boldsymbol{\alpha}_3 \\ \bar{\boldsymbol{\alpha}}_3 \\ 0 \\ 0 \\ \boldsymbol{\beta}_3 \\ \bar{\boldsymbol{\beta}}_3 \\ 0 \\ 0 \\ \boldsymbol{\gamma}_3 \\ \bar{\boldsymbol{\gamma}}_3 \\ 0 \\ 0 \end{matrix}\right\} + \begin{bmatrix} 0 \\ 0 \\ \mathbf{F}_3 \\ \mathbf{F}_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{q}_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mathbf{F}_3 \\ \mathbf{F}_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{q}_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mathbf{F}_3 \\ \mathbf{F}_4 \end{bmatrix} \mathbf{q}_4 \tag{32}$$

where $\mathbf{D}_{11}$, $\mathbf{D}_{12}$, $\mathbf{D}_{14}$, $\mathbf{D}_{18}$, $\mathbf{D}_{1\,10}$, $\mathbf{D}_{1\,12}$, $\mathbf{D}_{22}$, $\mathbf{D}_{23}$, $\mathbf{D}_{24}$, $\mathbf{D}_{27}$, $\mathbf{D}_{28}$, $\mathbf{D}_{2\,11}$, $\mathbf{D}_{2\,12}$, $\mathbf{D}_{54}$, $\mathbf{D}_{55}$, $\mathbf{D}_{56}$, $\mathbf{D}_{58}$, $\mathbf{D}_{5\,10}$, $\mathbf{D}_{5\,12}$, $\mathbf{D}_{63}$, $\mathbf{D}_{64}$, $\mathbf{D}_{66}$, $\mathbf{D}_{67}$, $\mathbf{D}_{68}$, $\mathbf{D}_{6\,11}$, $\mathbf{D}_{6\,12}$, $\mathbf{D}_{92}$, $\mathbf{D}_{96}$, $\mathbf{D}_{99}$, $\mathbf{D}_{9\,10}$, $\mathbf{D}_{9\,12}$, $\mathbf{D}_{10\,10}$, $\mathbf{D}_{10\,11}$ are 2×2 known square matrices including the values of the functions a_j, $\bar{a}_j$, β_j, $\bar{\beta}_j$, γ_j, $\bar{\gamma}_j$ $(j=1,2)$ of eqns (9)-(11); $\boldsymbol{\alpha}_3$, $\bar{\boldsymbol{\alpha}}_3$, $\boldsymbol{\beta}_3$, $\bar{\boldsymbol{\beta}}_3$, $\boldsymbol{\gamma}_3$, $\bar{\boldsymbol{\gamma}}_3$ are 2×1 known column matrices including the boundary values of the functions $a_3, \bar{a}_3, \beta_3, \bar{\beta}_3, \gamma_3, \bar{\gamma}_3$ of eqns (9)-(11); $\mathbf{E}_{jk}$, $(j=3,4,\ k=1,2,3,4)$ are square 2×2 known coefficient matrices resulting from the values of the kernels $\Lambda_j(r)$ $(j=1,2,3,4)$ at the beam ends and $\mathbf{F}_j$ $(j=3,4)$ are $2\times L$ rectangular known matrices originating from the integration of the kernels on the axis of the beam. Moreover,

$$\hat{\mathbf{u}}_i = \{u_i(0) \quad u_i(l)\}^T \quad \hat{\mathbf{u}}_{i,x} = \left\{\frac{du_i(0)}{dx} \quad \frac{du_i(l)}{dx}\right\}^T$$
$$\hat{\mathbf{u}}_{i,xx} = \left\{\frac{d^2u_i(0)}{dx^2} \quad \frac{d^2u_i(l)}{dx^2}\right\}^T \quad \hat{\mathbf{u}}_{i,xxx} = \left\{\frac{d^3u_i(0)}{dx^3} \quad \frac{d^3u_i(l)}{dx^3}\right\}^T \tag{33}$$

are vectors including the two unknown boundary values of the respective boundary quantities and $\mathbf{q}_i = \left\{q_1^i\ q_2^i\ \ldots\ q_L^i\right\}^T$ $(i=2,3,4)$ is the vector including the L unknown nodal values of the fictitious load.

Discretization of the integral representations of the displacement components u_i $(i=2,3,4)$ and their derivatives with respect to x [Sapountzakis and Dourakopoulos (2008)], after elimination of the boundary quantities employing eqns. (32), gives

$$\mathbf{u}_i = \mathbf{T}_i\mathbf{q}_i + \mathbf{T}_{ij}\mathbf{q}_j + \mathbf{T}_{ik}\mathbf{q}_k + \mathbf{t}_i \ i,j,k = 2,3,4\ i \neq j \neq k \tag{34a}$$

$$\mathbf{u}_{i,x} = \mathbf{T}_{ix}\mathbf{q}_i + \mathbf{T}_{ijx}\mathbf{q}_j + \mathbf{T}_{ikx}\mathbf{q}_k + \mathbf{t}_{ix} \ i,j,k = 2,3,4\ i \neq j \neq k \tag{34b}$$

$$\mathbf{u}_{i,xx} = \mathbf{T}_{ixx}\mathbf{q}_i + \mathbf{T}_{ijxx}\mathbf{q}_j + \mathbf{T}_{ikxx}\mathbf{q}_k + \mathbf{t}_{ixx} \ i,j,k = 2,3,4\ i \neq j \neq k \tag{34c}$$

$$\mathbf{u}_{i,xxx} = \mathbf{T}_{ixxx}\mathbf{q}_i + \mathbf{T}_{ijxxx}\mathbf{q}_j + \mathbf{T}_{ikxxx}\mathbf{q}_k + \mathbf{t}_{ixxx}\ i,j,k = 2,3,4\ i \neq j \neq k \tag{34d}$$

$$\mathbf{u}_{i,xxxx} = \mathbf{q}_i \quad i = 2,3,4 \tag{34e}$$

where $\mathbf{u}_i$, $\mathbf{u}_{i,x}$, $\mathbf{u}_{i,xx}$, $\mathbf{u}_{i,xxx}$, $\mathbf{u}_{i,xxxx}$ are vectors including the values of $u_i(x)$, $\mathbf{T}_i$, $\mathbf{T}_{ix}$, $\mathbf{T}_{ixx}$, $\mathbf{T}_{ixxx}$, $\mathbf{T}_{ij}$, $\mathbf{T}_{ijx}$, $\mathbf{T}_{ijxx}$, $\mathbf{T}_{ijxxx}$, $\mathbf{T}_{ik}$, $\mathbf{T}_{ikx}$, $\mathbf{T}_{ikxx}$, $\mathbf{T}_{ikxxx}$ are known $L \times L$ matrices and $\mathbf{t}_i$, $\mathbf{t}_{ix}$, $\mathbf{t}_{ixx}$, $\mathbf{t}_{ixxx}$ are known $L \times 1$ matrices.

In the conventional BEM, the load vectors $\mathbf{q}_i$ are known and eqns (34) are used to evaluate $u_i(x)$ and their derivatives at the L nodal points. This, however, can not be done here since $\mathbf{q}_i$ are unknown. For this purpose, $3L$ additional equations are derived, which permit the establishment of $\mathbf{q}_i$. These equations result by applying eqns (3) to the L collocation points, leading to the formulation of the following set of $3L$ simultaneous equations

$$(\mathbf{A} - \mathbf{NB} + \mathbf{C}) \begin{Bmatrix} \mathbf{q}_2 \\ \mathbf{q}_3 \\ \mathbf{q}_4 \end{Bmatrix} = \mathbf{f} \tag{35}$$

where the $3L \times 3L$ matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{N}$, $\mathbf{C}$ are given as

$$\mathbf{A} = \begin{bmatrix} \mathbf{EI}_{ZZ} & \mathbf{EI}_{YZ} & \mathbf{0} \\ \mathbf{EI}_{YZ} & \mathbf{EI}_{YY} & \mathbf{0} \\ -\mathbf{GI}_t\mathbf{T}_{42xx} & -\mathbf{GI}_t\mathbf{T}_{43xx} & \mathbf{EC}_S - \mathbf{GI}_t\mathbf{T}_{4xx} \end{bmatrix} \tag{36a}$$

$$\mathrm{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \mathbf{B}_{13} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \mathbf{B}_{23} \\ \mathbf{B}_{31} & \mathbf{B}_{32} & \mathbf{B}_{33} \end{bmatrix} \tag{36b}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13} \\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23} \\ \mathbf{C}_{31} & \mathbf{C}_{32} & \mathbf{C}_{33} \end{bmatrix} \tag{36c}$$

the $\mathbf{B}_{ij}$, $\mathbf{C}_{ij}$ $L \times L$ matrices are evaluated from the expressions

$$\mathbf{B}_{11} = -\frac{\alpha_Y}{G\mathrm{A}}\mathbf{EI}_Z + \mathbf{T}_{2xx} - z_C\mathbf{T}_{42xx} \tag{37a}$$

$$\mathbf{B}_{12} = -\frac{\alpha_Z}{G\mathrm{A}}\mathbf{EI}_{YZ} + \mathbf{T}_{23xx} - z_C\mathbf{T}_{43xx} \tag{37b}$$

$$\mathbf{B}_{13} = \frac{\alpha_Y z_C}{G\mathrm{A}}\mathbf{EI}_Z - \frac{\alpha_Z y_C}{G\mathrm{A}}\mathbf{EI}_{YZ} - z_C\mathbf{T}_{4xx} + \mathbf{T}_{24xx} \tag{37c}$$

$$\mathbf{B}_{21} = -\frac{\alpha_Y}{G\mathrm{A}}\mathbf{EI}_{YZ} + \mathbf{T}_{32xx} + y_C\mathbf{T}_{42xx} \tag{37d}$$

$$\mathbf{B}_{22} = -\frac{\alpha_Z}{G\mathrm{A}}\mathbf{EI}_Y + \mathbf{T}_{3xx} + y_C\mathbf{T}_{43xx} \tag{37e}$$

$$\mathbf{B}_{23} = \frac{\alpha_Y z_C}{G\mathrm{A}}\mathbf{EI}_{YZ} - \frac{\alpha_Z y_C}{G\mathrm{A}}\mathbf{EI}_Y + y_C\mathbf{T}_{4xx} + \mathbf{T}_{34xx} \tag{37f}$$

$$\mathbf{B}_{31} = y_C\mathbf{T}_{32xx} - z_C\mathbf{T}_{2xx} + \frac{I_S}{A}\mathbf{T}_{42xx} \tag{37g}$$

$$\mathbf{B}_{32} = y_C\mathbf{T}_{3xx} - z_C\mathbf{T}_{23xx} + \frac{I_S}{A}\mathbf{T}_{43xx} \tag{37h}$$

$$\mathbf{B}_{33} = \frac{I_S}{A}\mathbf{T}_{4xx} + y_C\mathbf{T}_{34xx} - z_C\mathbf{T}_{24xx} \tag{37i}$$

$$\begin{aligned}\mathbf{C}_{11} = \Big[&\mathbf{p}_X\mathbf{T}_{2x} - \frac{a_Y}{GA}\mathbf{EI}_Z\left(\mathbf{p}_{X,xx}\mathbf{T}_{2x} + 3\mathbf{p}_{X,x}\mathbf{T}_{2xx} + 3\mathbf{p}_X\mathbf{T}_{2xxx}\right) - \\ &-\frac{a_Z}{GA}\mathbf{EI}_{YZ}\left(\mathbf{p}_{X,xx}\mathbf{T}_{32x} + 3\mathbf{p}_{X,x}\mathbf{T}_{32xx} + 3\mathbf{p}_X\mathbf{T}_{32xxx}\right) - \\ &- z_C\mathbf{p}_X\mathbf{T}_{42x} + \frac{a_Y}{GA}\mathbf{EI}_Z\left(z_C\mathbf{p}_{X,xx}\mathbf{T}_{42x} + 3z_C\mathbf{p}_{X,x}\mathbf{T}_{42xx} + 3z_C\mathbf{p}_X\mathbf{T}_{42xxx}\right) - \\ &-\frac{a_Z}{GA}\mathbf{EI}_{YZ}\left(y_C\mathbf{p}_{X,xx}\mathbf{T}_{42x} + 3y_C\mathbf{p}_{X,x}\mathbf{T}_{42xx} + 3y_C\mathbf{p}_X\mathbf{T}_{42xxx}\right)\Big]\end{aligned} \tag{38a}$$

$$\begin{aligned}\mathbf{C}_{12} = \Big[&\mathbf{p}_X\mathbf{T}_{23x} - \frac{a_Y}{GA}\mathbf{EI}_Z\left(\mathbf{p}_{X,xx}\mathbf{T}_{23x} + 3\mathbf{p}_{X,x}\mathbf{T}_{23xx} + 3\mathbf{p}_X\mathbf{T}_{23xxx}\right) - \\ &-\frac{a_Z}{GA}\mathbf{EI}_{YZ}\left(\mathbf{p}_{X,xx}\mathbf{T}_{3x} + 3\mathbf{p}_{X,x}\mathbf{T}_{3xx} + 3\mathbf{p}_X\mathbf{T}_{3xxx}\right) - \\ &- z_C\mathbf{p}_X\mathbf{T}_{43x} + \frac{a_Y}{GA}\mathbf{EI}_Z\left(z_C\mathbf{p}_{X,xx}\mathbf{T}_{43x} + 3z_C\mathbf{p}_{X,x}\mathbf{T}_{43xx} + 3z_C\mathbf{p}_X\mathbf{T}_{43xxx}\right) - \\ &-\frac{a_Z}{GA}\mathbf{EI}_{YZ}\left(y_C\mathbf{p}_{X,xx}\mathbf{T}_{43x} + 3y_C\mathbf{p}_{X,x}\mathbf{T}_{43xx} + 3y_C\mathbf{p}_X\mathbf{T}_{43xxx}\right)\Big]\end{aligned} \tag{38b}$$

$$\begin{aligned}\mathbf{C}_{13} = \Big[&\mathbf{p}_X\mathbf{T}_{24x} - \frac{a_Y}{GA}\mathbf{EI}_Z\left(\mathbf{p}_{X,xx}\mathbf{T}_{24x} + 3\mathbf{p}_{X,x}\mathbf{T}_{24xx} + 3\mathbf{p}_X\mathbf{T}_{24xxx}\right) - \\ &-\frac{a_Z}{GA}\mathbf{EI}_{YZ}\left(\mathbf{p}_{X,xx}\mathbf{T}_{34x} + 3\mathbf{p}_{X,x}\mathbf{T}_{34xx} + 3\mathbf{p}_X\mathbf{T}_{34xxx}\right) - \\ &- z_C\mathbf{p}_X\mathbf{T}_{4x} + \frac{a_Y}{GA}\mathbf{EI}_Z\left(z_C\mathbf{p}_{X,xx}\mathbf{T}_{4x} + 3z_C\mathbf{p}_{X,x}\mathbf{T}_{4xx} + 3z_C\mathbf{p}_X\mathbf{T}_{4xxx}\right) - \\ &-\frac{a_Z}{GA}\mathbf{EI}_{YZ}\left(y_C\mathbf{p}_{X,xx}\mathbf{T}_{4x} + 3y_C\mathbf{p}_{X,x}\mathbf{T}_{4xx} + 3y_C\mathbf{p}_X\mathbf{T}_{4xxx}\right)\Big]\end{aligned} \tag{38c}$$

$$\begin{aligned}\mathbf{C}_{21} = \Big[&y_C\mathbf{p}_X\mathbf{T}_{42x} - \frac{a_Z}{GA}\mathbf{EI}_Y\left(y_C\mathbf{p}_{X,xx}\mathbf{T}_{42x} + 3y_C\mathbf{p}_{X,x}\mathbf{T}_{42xx} + 3y_C\mathbf{p}_X\mathbf{T}_{42xxx}\right) + \\ &+\frac{a_Y}{GA}\mathbf{EI}_{YZ}\left(z_C\mathbf{p}_{X,xx}\mathbf{T}_{42x} + 3z_C\mathbf{p}_{X,x}\mathbf{T}_{42xx} + 3z_C\mathbf{p}_X\mathbf{T}_{42xxx}\right) + \\ &+ \mathbf{p}_X\mathbf{T}_{32x} - \frac{a_Z}{GA}\mathbf{EI}_Y\left(\mathbf{p}_{X,xx}\mathbf{T}_{32x} + 3\mathbf{p}_{X,x}\mathbf{T}_{32xx} + 3\mathbf{p}_X\mathbf{T}_{32xxx}\right) - \\ &-\frac{a_Y}{GA}\mathbf{EI}_{YZ}\left(\mathbf{p}_{X,xx}\mathbf{T}_{2x} + 3\mathbf{p}_{X,x}\mathbf{T}_{2xx} + 3\mathbf{p}_X\mathbf{T}_{2xxx}\right)\Big]\end{aligned} \tag{38d}$$

$$
\begin{aligned}
\mathbf{C}_{22} = \Big[& y_C \mathbf{p}_X \mathbf{T}_{43x} - \frac{a_Z}{GA} \mathbf{EI}_Y \left(y_C \mathbf{p}_{X,xx} \mathbf{T}_{43x} + 3 y_C \mathbf{p}_{X,x} \mathbf{T}_{43xx} + 3 y_C \mathbf{p}_X \mathbf{T}_{43xxx} \right) + \\
& + \frac{a_Y}{GA} \mathbf{EI}_{YZ} \left(z_C \mathbf{p}_{X,xx} \mathbf{T}_{43x} + 3 z_C \mathbf{p}_{X,x} \mathbf{T}_{43xx} + 3 z_C \mathbf{p}_X \mathbf{T}_{43xxx} \right) + \\
& + \mathbf{p}_X \mathbf{T}_{3x} - \frac{a_Z}{GA} \mathbf{EI}_Y \left(\mathbf{p}_{X,xx} \mathbf{T}_{3x} + 3 \mathbf{p}_{X,x} \mathbf{T}_{3xx} + 3 \mathbf{p}_X \mathbf{T}_{3xxx} \right) - \\
& - \frac{a_Y}{GA} \mathbf{EI}_{YZ} \left(\mathbf{p}_{X,xx} \mathbf{T}_{23x} + 3 \mathbf{p}_{X,x} \mathbf{T}_{23xx} + 3 \mathbf{p}_X \mathbf{T}_{23xxx} \right) \Big]
\end{aligned} \tag{38e}
$$

$$
\begin{aligned}
\mathbf{C}_{23} = \Big[& y_C \mathbf{p}_X \mathbf{T}_{4x} - \frac{a_Z}{GA} \mathbf{EI}_Y \left(y_C \mathbf{p}_{X,xx} \mathbf{T}_{4x} + 3 y_C \mathbf{p}_{X,x} \mathbf{T}_{4xx} + 3 y_C \mathbf{p}_X \mathbf{T}_{4xxx} \right) + \\
& + \frac{a_Y}{GA} \mathbf{EI}_{YZ} \left(z_C \mathbf{p}_{X,xx} \mathbf{T}_{4x} + 3 z_C \mathbf{p}_{X,x} \mathbf{T}_{4xx} + 3 z_C \mathbf{p}_X \mathbf{T}_{4xxx} \right) + \\
& + \mathbf{p}_X \mathbf{T}_{34x} - \frac{a_Z}{GA} \mathbf{EI}_Y \left(\mathbf{p}_{X,xx} \mathbf{T}_{34x} + 3 \mathbf{p}_{X,x} \mathbf{T}_{34xx} + 3 \mathbf{p}_X \mathbf{T}_{34xxx} \right) - \\
& - \frac{a_Y}{GA} \mathbf{EI}_{YZ} \left(\mathbf{p}_{X,xx} \mathbf{T}_{24x} + 3 \mathbf{p}_{X,x} \mathbf{T}_{24xx} + 3 \mathbf{p}_X \mathbf{T}_{24xxx} \right) \Big]
\end{aligned} \tag{38f}
$$

$$
\mathbf{C}_{31} = \left[y_C \mathbf{p}_X \mathbf{T}_{32x} - z_C \mathbf{p}_X \mathbf{T}_{2x} + \frac{I_S}{A} \mathbf{p}_X \mathbf{T}_{42x} \right] \tag{38g}
$$

$$
\mathbf{C}_{32} = \left[y_C \mathbf{p}_X \mathbf{T}_{3x} - z_C \mathbf{p}_X \mathbf{T}_{23x} \frac{I_S}{A} \mathbf{p}_X \mathbf{T}_{43x} \right] \tag{38h}
$$

$$
\mathbf{C}_{33} = \left[\frac{I_S}{A} \mathbf{p}_X \mathbf{T}_{4x} + y_C \mathbf{p}_X \mathbf{T}_{34x} - z_C \mathbf{p}_X \mathbf{T}_{24x} \right] \tag{38i}
$$

$$
\mathbf{N} = \begin{bmatrix}
N_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & N_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & N_L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & N_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & N_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & N_L & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_L
\end{bmatrix} \tag{39}
$$

and the $3L \times 1$ column matrix $\mathbf{f}$ is given as

$$
\mathbf{f} = \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ GI_t \mathbf{t}_{4xx} \end{Bmatrix} + \mathbf{N} \begin{Bmatrix} \mathbf{t}_{2xx} - z_C \mathbf{t}_{4xx} \\ \mathbf{t}_{3xx} + y_C \mathbf{t}_{4xx} \\ y_C \mathbf{t}_{3xx} - z_C \mathbf{t}_{2xx} + \frac{I_S}{A} \mathbf{t}_{4xx} \end{Bmatrix} \tag{40}
$$

with

$$\begin{aligned}\mathbf{f}_1 =&\mathbf{p}_Y - \mathbf{p}_X\left(\mathbf{t}_{2x} - z_C\mathbf{t}_{4x}\right) - \frac{a_Y}{GA}\mathbf{EI}_{ZZ}\left(\mathbf{p}_{Y,xx} - \mathbf{p}_{X,xx}\left(\mathbf{t}_{2x} - z_C\mathbf{t}_{4x}\right) -\right.\\ &-3\mathbf{p}_{X,x}\left(\mathbf{t}_{2xx} - z_C\mathbf{t}_{4xx}\right) - 3\mathbf{p}_X\left(\mathbf{t}_{2xxx} - z_C\mathbf{t}_{4xxx}\right)\Big) - \frac{a_Z}{GA}\mathbf{EI}_{\mathbf{YZ}}\left(\mathbf{p}_{Z,xx}-\right.\\ &\left.- \mathbf{p}_{X,xx}\left(\mathbf{t}_{3x} + y_C\mathbf{t}_{4x}\right) - 3\mathbf{p}_{X,x}\left(\mathbf{t}_{3xx} + y_C\mathbf{t}_{4xx}\right) - 3\mathbf{p}_X\left(\mathbf{t}_{3xxx} + y_C\mathbf{t}_{4xxx}\right)\right) - \mathbf{m}_{Z,x}\end{aligned} \tag{41a}$$

$$\begin{aligned}\mathbf{f}_2 =&\mathbf{p}_Z - \mathbf{p}_X\left(\mathbf{t}_{3x} + y_C\mathbf{t}_{4x}\right) - \frac{a_Z}{GA}\mathbf{EI}_{YY}\left(\mathbf{p}_{Z,xx} - \mathbf{p}_{X,xx}\left(\mathbf{t}_{3x} + y_C\mathbf{t}_{4x}\right) -\right.\\ &-3\mathbf{p}_{X,x}\left(\mathbf{t}_{3xx} + y_C\mathbf{t}_{4xx}\right) - 3\mathbf{p}_X\left(\mathbf{t}_{3xxx} + y_C\mathbf{t}_{4xxx}\right)\Big) - \frac{a_Y}{GA}\mathbf{EI}_{\mathbf{YZ}}\left(\mathbf{p}_{Y,xx} -\right.\\ &\left.-\mathbf{p}_{X,xx}\left(\mathbf{t}_{2x} - z_C\mathbf{t}_{4x}\right) - 3\mathbf{p}_{X,x}\left(\mathbf{t}_{2xx} - z_C\mathbf{t}_{4xx}\right) - 3\mathbf{p}_X\left(\mathbf{t}_{2xxx} - z_C\mathbf{t}_{4xxx}\right)\right) + \mathbf{m}_{Y,x}\end{aligned} \tag{41b}$$

$$\mathbf{f}_3 = \mathbf{m}_x + \mathbf{p}_Z y_C - \mathbf{p}_Y z_C + z_C\mathbf{p}_X\mathbf{t}_{2x} - y_C\mathbf{p}_X\mathbf{t}_{3x} - \frac{I_S}{A}\mathbf{p}_X\mathbf{t}_{4x} \tag{41c}$$

In the above set of equations the matrices $\mathbf{EI}_{YY}$, $\mathbf{EI}_{ZZ}$, $\mathbf{EI}_{YZ}$, $\mathbf{EC}_S$, $\mathbf{GI}_t$ are $L\times L$ diagonal matrices including the values of the corresponding quantities, respectively, at the L nodal points. Moreover, $\mathbf{p}_X$, $\mathbf{p}_{X,x}$, $\mathbf{p}_{X,xx}$ are diagonal matrices and $\mathbf{p}_Y$, $\mathbf{p}_{Y,xx}$, $\mathbf{p}_Z$, $\mathbf{p}_{Z,xx}$, $\mathbf{m}_{Y,x}\mathbf{m}_{Z,x}$ and $\mathbf{m}_x$are vectors containing the values of the external loading and their derivatives at these points.

Solving the nonlinear system of eqns (35) for the fictitious load distributions $\mathbf{q}_2$, $\mathbf{q}_3$, $\mathbf{q}_4$ the displacements and their derivatives in the interior of the beam are computed using eqns (34).

3.2 For the axial displacement u

Let $u_1 = u$ be the sought solution of the boundary value problem described by eqns (21) and (22). Differentiating this function two times yields

$$\frac{d^2u_1}{dx^2} = q_1\left(x\right) \tag{42}$$

Eqn (42) indicates that the solution of the original problem can be obtained as the axial displacement of a beam-column with unit axial rigidity subjected to an axial fictitious load $q_1\left(x\right)$ under the same boundary conditions. The fictitious load is unknown.

Following the same procedure as in 3.1, the integral representation of the displacement components u_1 and its derivatives with respect to x when applied to all nodal points in the interior of the beam-column yields

$$\mathbf{u}_1 = \mathbf{T}_1\mathbf{q}_1 + \mathbf{t}_1 \quad \mathbf{u}_{1,x} = \mathbf{T}_{1x}\mathbf{q}_1 + \mathbf{t}_{1x} \quad \mathbf{u}_{1,xx} = \mathbf{q}_1 \tag{43}$$

where $\mathbf{T}_1$, $\mathbf{T}_{1x}$ are known matrices with dimensions LxL, similar with those mentioned before for the displacements u_2, u_3, u_4 and the following system of equations with respect to $\mathbf{q_1}$, $\mathbf{q_2}$, $\mathbf{q_3}$ and $\mathbf{q_4}$ is obtained

$$\begin{aligned}
&\mathbf{EAq_1}+\\
&+\mathbf{EA}\left[\left(\mathbf{T}_{2xx}\mathbf{q}_2+\mathbf{T}_{23xx}\mathbf{q}_3+\mathbf{T}_{24xx}\mathbf{q}_4+\mathbf{t}_{2xx}\right)\right]_{dg}\left(\mathbf{T}_{2x}\mathbf{q}_2+\mathbf{T}_{23x}\mathbf{q}_3+\mathbf{T}_{24x}\mathbf{q}_4+\mathbf{t}_{2x}\right)+\\
&+\mathbf{EA}\left[\left(\mathbf{T}_{3xx}\mathbf{q}_3+\mathbf{T}_{32xx}\mathbf{q}_2+\mathbf{T}_{34xx}\mathbf{q}_4+\mathbf{t}_{3xx}\right)\right]_{dg}\left(\mathbf{T}_{3x}\mathbf{q}_3+\mathbf{T}_{32x}\mathbf{q}_2+\mathbf{T}_{34x}\mathbf{q}_4+\mathbf{t}_{3x}\right)=\\
&=-\mathbf{p_x}
\end{aligned} \tag{44}$$

In the above set of equations the matrix $\mathbf{EA}$ is an $L\times L$ diagonal matrix including the values of the corresponding quantities, respectively, at the L nodal points, while the axial force following eqn.(20) can be expressed as

$$\begin{aligned}
\mathrm{N}=&\frac{1}{2}\mathbf{EA}\left(\mathbf{T}_{1x}\mathbf{q}_1+\mathbf{t}_{1x}\right)\\
&+\frac{1}{2}\mathbf{EA}\left[\left(\mathbf{T}_{2x}\mathbf{q}_2+\mathbf{T}_{23x}\mathbf{q}_3+\mathbf{T}_{24x}\mathbf{q}_4+\mathbf{t}_{2x}\right)\right]_{dg}\left(\mathbf{T}_{2x}\mathbf{q}_2+\mathbf{T}_{23x}\mathbf{q}_3+\mathbf{T}_{24x}\mathbf{q}_4+\mathbf{t}_{2x}\right)\\
&+\frac{1}{2}\mathbf{EA}\left[\left(\mathbf{T}_{3x}\mathbf{q}_3+\mathbf{T}_{32x}\mathbf{q}_2+\mathbf{T}_{34x}\mathbf{q}_4+\mathbf{t}_{3x}\right)\right]_{dg}\left(\mathbf{T}_{3x}\mathbf{q}_3+\mathbf{T}_{32x}\mathbf{q}_2+\mathbf{T}_{34x}\mathbf{q}_4+\mathbf{t}_{3x}\right)
\end{aligned} \tag{45}$$

Eqns. (35), (44) and (45) constitute a nonlinear coupled system of equations with respect to $\mathbf{q_1}$, $\mathbf{q_2}$, $\mathbf{q_3}$, $\mathbf{q_4}$ and N quantities. The solution of this system is accomplished iteratively by employing the two term acceleration method [Isaacson and Keller (1966), Sapountzakis and Katsikadelis (1992)].

3.3 For the primary warping function φ_S^P

The numerical solution for the evaluation of the displacement and rotation components assume that the warping C_S and torsion I_t constants given from eqns (5), (6) are already established. Eqns (5), (6) indicate that the evaluation of the aforementioned constants presumes that the primary warping function φ_S^P at any interior point of the domain Ω of the cross section of the beam is known. Once φ_S^P is established, C_S and I_t constants are evaluated by converting the domain integrals into line integrals along the boundary as this is presented in Sapountzakis [Sapountzakis (2000), Sapountzakis (2001)] and in Sapountzakis and Mokos [Sapountzakis and Mokos (2003)].

3.4. For the stress functions $\Theta(Y,Z)$ and $\Phi(Y,Z)$

The evaluation of the stress functions $\Theta(Y,Z)$ and $\Phi(Y,Z)$ is accomplished using BEM as this is presented in Sapountzakis and Mokos [Sapountzakis and Mokos (2005)].

4 Numerical examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the efficiency, the accuracy and the range of applications of the developed method. In all the examples treated, the results have been obtained using $L = 51$ nodal points along the beam.

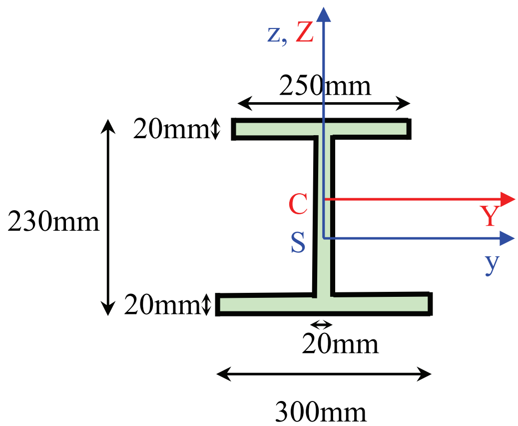

Figure 2: I-shaped cross section of the clamped beam of Example 1.

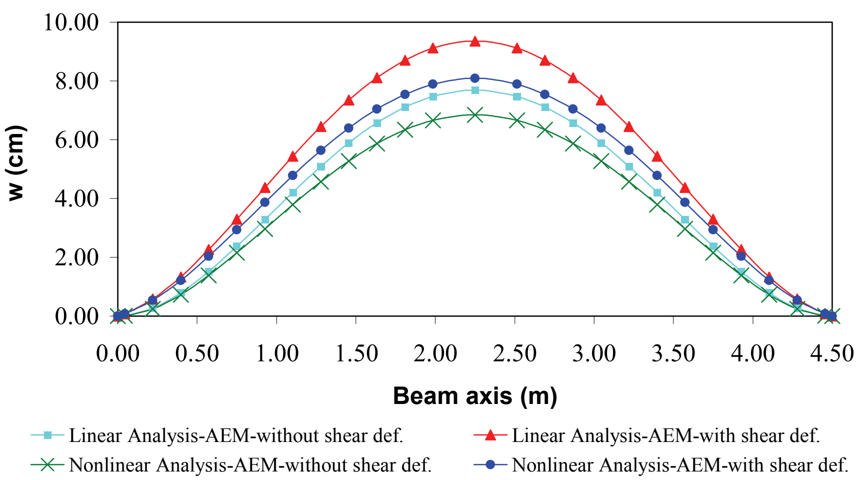

Figure 3: Displacement w of the beam of example 1 for $p_Y = p_Z = 2000kN/m$.

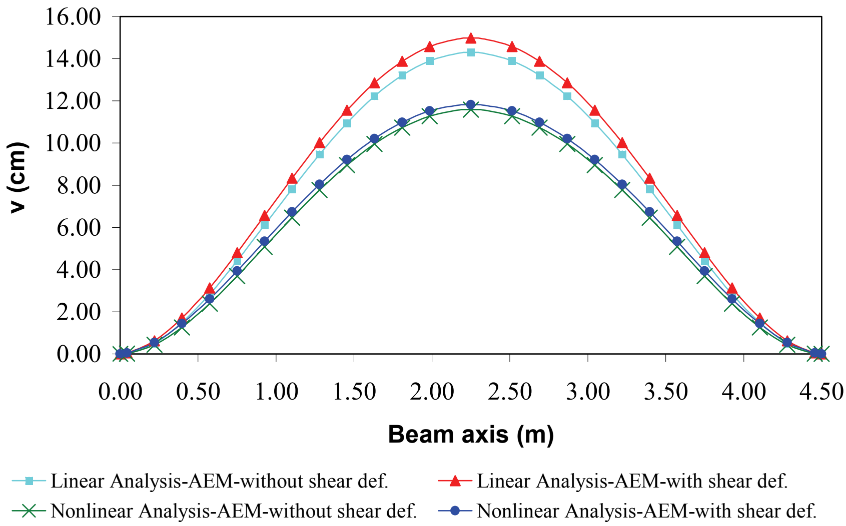

Figure 4: Displacement v of the beam of example 1 for $p_Y = p_Z = 2000 kN/m$.

Table 1: Displacement w at the midspan of the clamped beam of example 1 for various values of the transverse loading $p_Y = p_Z$.

	Displacement w (cm)			
p_Y, p_Z	Linear Analysis		Nonlinear Analysis	
	Ignoring Shear Deformation ($a_Z = 0.0$)	With Shear Deformation ($a_Z = 3.932$)	Ignoring Shear Deformation ($a_Z = 0.0$)	With Shear Deformation ($a_Z = 3.932$)
500	1.921	2.338	1.901	2.305
1000	3.843	4.675	3.697	4.446
1500	5.764	7.013	5.345	6.368
2000	7.686	9.351	6.844	8.083
2500	9.607	11.688	8.213	9.625
3000	11.528	14.026	9.470	11.024
3500	13.450	16.364	10.632	12.304

4.1 Example 1

A clamped beam of length $l = 4.50m$ ($E = 2.1 \times 10^8 kN/m^2$, $v = 0.3$) of the monosymmetric I-shaped cross section of Fig.2 ($A = 1.48 \times 10^{-2} m^2$, $I_{YY} = 1.323 \times$

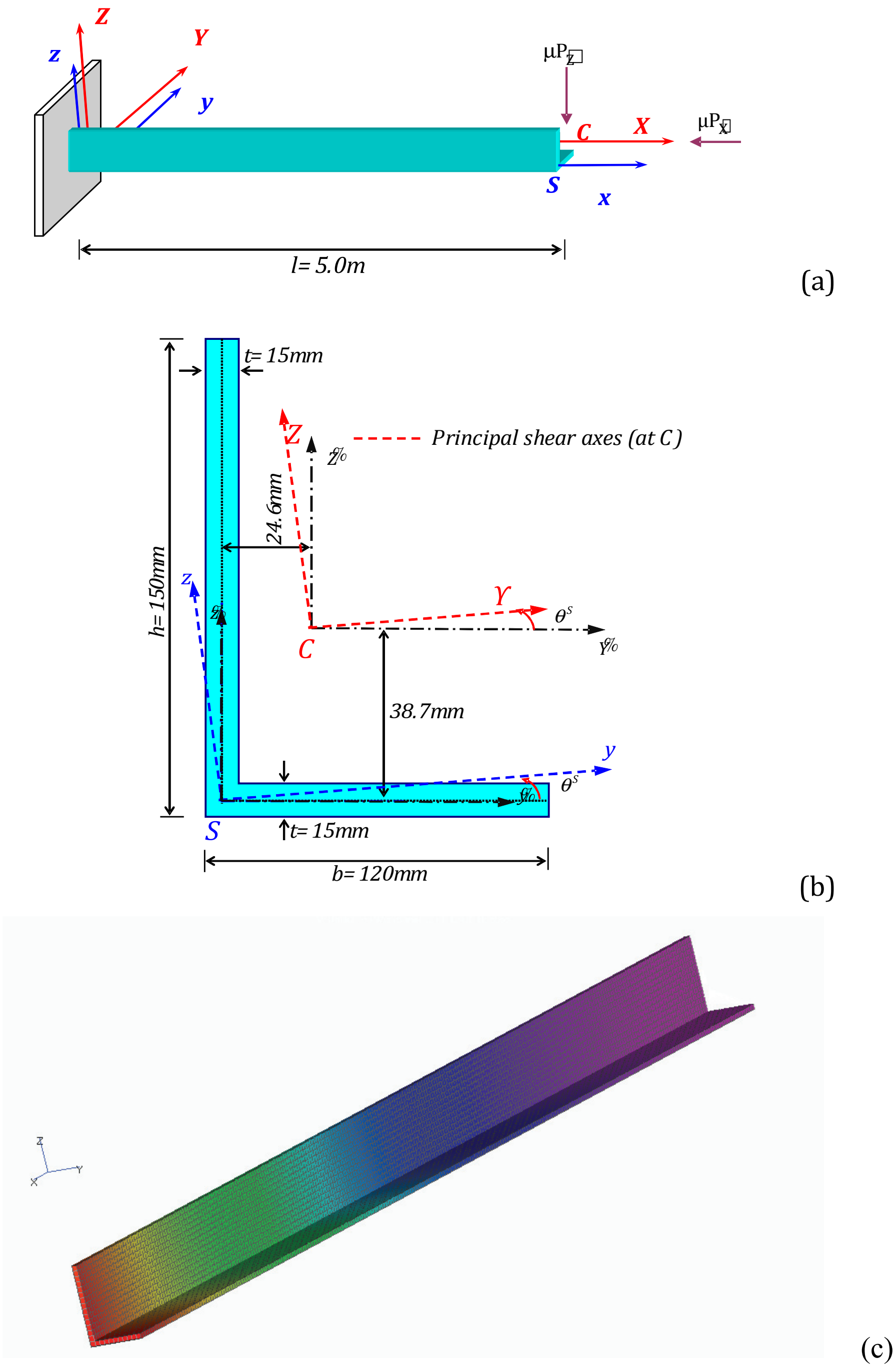

Figure 5: 3-D view (a), cross section (b) and 3-D FEM model (c) of the L-shaped cantilever beam of Example 2.

Table 2: Displacement v at the midspan of the clamped beam of example 1 for various values of the transverse loading $p_Y = p_Z$.

	Displacement v (cm)			
p_Y, p_Z	Linear Analysis		Nonlinear Analysis	
	Ignoring Shear Deformation ($a_Y = 0.0$)	With Shear Deformation ($a_Y = 1.629$)	Ignoring Shear Deformation ($a_Y = 0.0$)	With Shear Deformation ($a_Y = 1.629$)
500	3.573	3.745	3.500	3.655
1000	7.145	7.490	6.647	6.884
1500	10.718	11.235	9.325	9.579
2000	14.290	14.980	11.584	11.822
2500	17.863	18.725	13.513	13.721
3000	21.436	22.470	15.186	15.362
3500	25.008	26.215	16.660	16.806

Table 3: Displacement w at the midspan of the clamped beam of example 1 for various numbers of the beam collocation points.

	Displacement w (cm)			
p_Y, p_Z	Number of internal beam elements			
	11 elements	31 elements	51 elements	71 elements
500	2.306	2.305	2.305	2.305
1000	4.454	4.447	4.446	4.446
1500	6.388	6.369	6.368	6.367
2000	8.118	8.085	8.083	8.082
2500	9.675	9.628	9.625	9.624
3000	11.090	11.028	11.024	11.022
3500	12.386	12.310	12.304	12.303

Table 4: Displacement v at the midspan of the clamped beam of example 1 for various numbers of the beam collocation points.

	Displacement v (cm)			
p_Y, p_Z	Number of internal beam elements			
	11 elements	31 elements	51 elements	71 elements
500	3.660	3.656	3.655	3.655
1000	6.909	6.886	6.884	6.883
1500	9.638	9.583	9.579	9.577
2000	11.918	11.829	11.822	11.820
2500	13.854	13.731	13.721	13.719
3000	15.528	15.375	15.362	15.359
3500	17.003	16.820	16.806	16.802

Table 5: Geometric, inertia constants and shear deformation coefficients of the cross section of example 2.

Coordinate system $C\tilde{Y}\tilde{Z}$	Coordinate system CYZ
$I_{\tilde{Y}\tilde{Y}} = 8.470 \times 10^{-6} m^4$	$I_{YY} = 8.639 \times 10^{-6} m^4$
$I_{\tilde{Z}\tilde{Z}} = 4.825 \times 10^{-6} m^4$	$I_{ZZ} = 4.655 \times 10^{-6} m^4$
$I_{\tilde{Y}\tilde{Z}} = -3.752 \times 10^{-6} m^4$	$I_{YZ} = -3.665 \times 10^{-6} m^4$
$\alpha_{\tilde{Y}} = 2.626$	$\alpha_Y = 2.627$
$\alpha_{\tilde{Z}} = 2.018$	$\alpha_Z = 2.017$
$\alpha_{\tilde{Y}\tilde{Z}} = 0.014$	$\alpha_{YZ} = 0.0$
$\tilde{y}_C = 2.46 \times 10^{-2} m$	$y_C = 2.54 \times 10^{-2} m$
$\tilde{z}_C = 3.87 \times 10^{-2} m$	$z_C = 3.81 \times 10^{-2} m$
$\theta^S = 0.023 rad$	-

Table 6: Displacement w at the free end of the cantilever beam of example 2 for various values of the loading factor μ.

Scale factor μ	Displacement w (cm)				
	Linear Analysis		Nonlinear Analysis		
	Ignoring Shear Deformation ($a_Z = 0$)	With Shear Deformation ($a_Z = 2.017$)	Ignoring Shear Deformation ($a_Z = 0$)	With Shear Deformation ($a_Z = 2.017$)	FEM [MSC NASTRAN for Windows]
1	-0.16	-0.16	-0.16	-0.16	-0.16
2	-0.31	-0.31	-0.34	-0.34	-0.35
3	-0.47	-0.47	-0.53	-0.53	-0.54
4	-0.62	-0.62	-0.75	-0.75	-0.77
5	-0.78	-0.78	-0.99	-0.99	-1.01
6	-0.93	-0.93	-1.26	-1.26	-1.29
7	-1.09	-1.09	-1.58	-1.58	-1.62
8	-1.24	-1.24	-1.94	-1.95	-2.00
9	-1.40	-1.40	-2.38	-2.39	-2.45
10	-1.56	-1.56	-2.92	-2.93	-3.00

Table 7: Displacement v at the free end of the cantilever beam of example 2 for various values of the loading factor μ.

Scale factor μ	Displacement v (cm)				
	Linear Analysis		Nonlinear Analysis		
	Ignoring Shear Deformation ($a_Y = 0$)	With Shear Deformation ($a_Y = 2.627$)	Ignoring Shear Deformation ($a_Y = 0$)	With Shear Deformation ($a_Y = 2.627$)	FEM [MSC NASTRAN for Windows]
1	-0.13	-0.13	-0.14	-0.14	-0.14
2	-0.25	-0.25	-0.29	-0.29	-0.30
3	-0.38	-0.38	-0.48	-0.48	-0.48
4	-0.51	-0.51	-0.69	-0.69	-0.70
5	-0.63	-0.63	-0.94	-0.94	-0.96
6	-0.76	-0.76	-1.25	-1.25	-1.27
7	-0.89	-0.89	-1.61	-1.61	-1.65
8	-1.01	-1.01	-2.06	-2.07	-2.12
9	-1.14	-1.14	-2.63	-2.64	-2.72
10	-1.27	-1.27	-3.36	-3.37	-3.47

Table 8: Geometric, inertia constants and shear deformation coefficients of the cross section of example 3.

Coordinate system $C\tilde{Y}\tilde{Z}$	Coordinate system CYZ
$I_{\tilde{Y}\tilde{Y}} = 1.606 \times 10^{-4} m^4$	$I_{YY} = 1.545 \times 10^{-4} m^4$
$I_{\tilde{Z}\tilde{Z}} = 5.665 \times 10^{-5} m^4$	$I_{ZZ} = 6.278 \times 10^{-5} m^4$
$I_{\tilde{Y}\tilde{Z}} = 6.384 \times 10^{-5} m^4$	$I_{YZ} = 6.837 \times 10^{-5} m^4$
$\alpha_{\tilde{Y}} = 1.741$	$\alpha_Y = 1.736$
$\alpha_{\tilde{Z}} = 3.902$	$\alpha_Z = 3.907$
$\alpha_{\tilde{Y}\tilde{Z}} = -0.10$	$\alpha_{YZ} = 0.0$
$\tilde{y}_C = -3.84 \times 10^{-2} m$	$y_C = -4.01 \times 10^{-2} m$
$\tilde{z}_C = -3.79 \times 10^{-2} m$	$z_C = -3.61 \times 10^{-2} m$
$\theta^S = 0.046 rad$	-

Table 9: Displacement w at the free end of the cantilever beam of example 3 for various values of the loading factor μ.

Scale factor μ	Displacement w (cm)			
	Linear Analysis		Nonlinear Analysis	
	Ignoring Shear Deformation	With Shear Deformation ($a_Z = 3.907$)	Ignoring Shear Deformation	With Shear Deformation ($a_Z = 3.907$)
1	-1.10	-1.20	-1.15	-1.26
2	-2.20	-2.41	-2.42	-2.63
3	-3.31	-3.61	-3.83	-4.16
4	-4.41	-4.82	-5.41	-5.87
5	-5.51	-6.02	-7.21	-7.81
6	-6.61	-7.22	-9.29	-10.05
7	-7.72	-8.43	-11.75	-12.69
8	-8.82	-9.63	-14.72	-15.88
9	-9.92	-10.84	-18.42	-19.86
10	-11.02	-12.04	-23.23	-25.04

Scale factor μ	Displacement v (cm)			
	Linear Analysis		Nonlinear Analysis	
	Ignoring Shear Deformation	With Shear Deformation ($a_Y = 1.736$)	Ignoring Shear Deformation	With Shear Deformation ($a_Y = 1.736$)
1	1.13	1.13	1.22	1.22
2	2.26	2.26	2.63	2.64
3	3.40	3.40	4.30	4.32
4	4.53	4.53	6.26	6.31
5	5.66	5.66	8.62	8.71
6	6.79	6.79	11.49	11.64
7	7.92	7.92	15.05	15.29
8	9.05	9.05	19.54	19.93
9	10.19	10.18	25.38	26.00
10	11.32	11.31	33.26	34.26

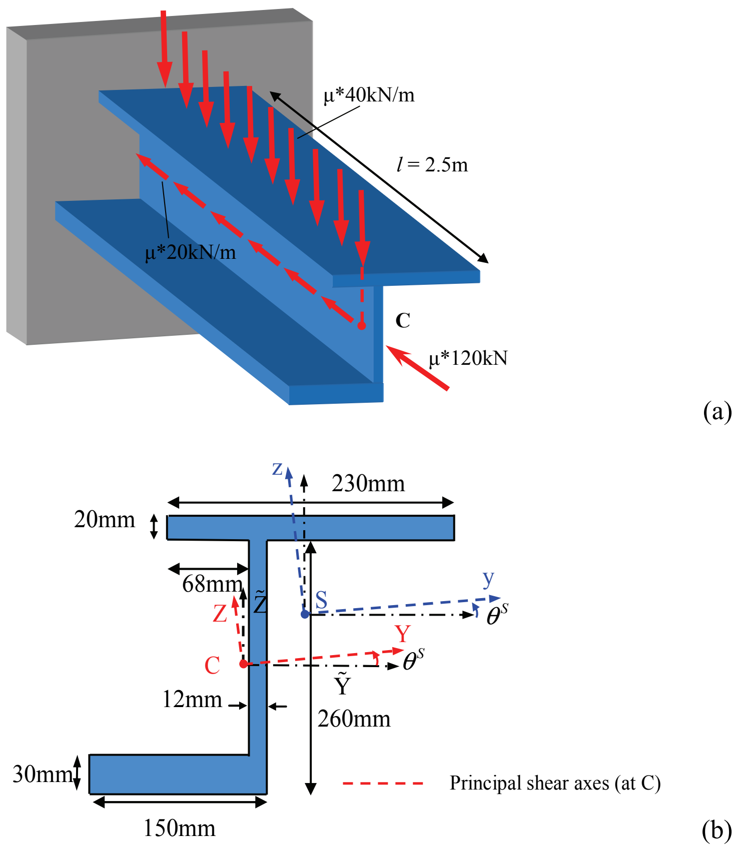

Figure 6: 3-D view (a) and cross section (b) of the non-symmetric beam of Example 3.

$10^{-4}m^4$, $I_{ZZ} = 7.117 \times 10^{-5}m^4$, $C_s = 7.225 \times 10^{-7}m^6$, $I_t = 2.00 \times 10^{-6}m^4$, $a_Y = 1.629$, $a_Z = 3.932$, $z_C = 2.07 \times 10^{-2}m$) subjected to the transverse uniformly distributed loading p_Y=p_Z acting at the centre of gravity in the Y and Z directions, has been studied. The displacements w, v in Tables 1, 2 at the midspan of the beam for various values of the transverse loading and in Figs. 3, 4 along the beam shear center axis for $p_Y = p_Z = 2000kN/m$ are presented as compared with those obtained from a linear analysis taking into account or ignoring shear deformation effect. From these tables and figures the significant influence of the nonlinear analysis effect is easily verified especially in the case of intense loading. Moreover, the influence of the shear deformation effect to the displacement v is not significant (low value of the shear deformation coefficient a_Y) and could be ignored, which does not hold for the displacement w(higher value of the shear deformation coefficient a_Z and bending stiffness EI_{YY}), where their remarkable increment due to the aforementioned effect necessitates its inclusion in the beam analysis. Finally, in Tables 3, 4 the displacement components w, v obtained from the nonlinear analysis of Timoshenko beams are presented for various values of the transverse loading and for various discretization schemes, demonstrating the fast convergence of the proposed method for a small number of collocation points.

4.2 Example 2

In the second example, in order to demonstrate the effectiveness of the proposed method, a cantilever beam of length $l = 5.0m$ (Fig.5a), of the non-symmetric L-shaped cross section of unequal legs ($A = 3.825 \times 10^{-3} m^2$, $E = 2.1 \times 10^8 kN/m^2$, $\nu = 0.3$, $C_s = 3.946 \times 10^{-10} m^6$, $I_t = 2.797 \times 10^{-7} m^4$) of Fig. 5b has been studied. Since the proposed analysis has been developed with respect to the principal shear system of axes, the geometric, the inertia constants and the shear deformation coefficients of the examined cross section with respect to an original coordinate system $C\tilde{Y}\tilde{Z}$ (Fig. 5b) are presented in the first column of Table 5, while in the second column of this table the same constants are evaluated with respect to axes parallel to the principal shear coordinate system CYZ, resulting from the rotation of the $C\tilde{Y}\tilde{Z}$ system by the angle θ^S [Sapountzakis and Mokos (2005)]. In Tables 6, 7 the displacements w, v at the free end of the cantilever beam, subjected to a gradually increasing concentrated transverse $\mu P_{\tilde{Z}}$ and axial $\mu P_{\tilde{X}}$ load ($P_{\tilde{Z}} = -0.045kN$, $P_{\tilde{X}} = -3.00kN$) at the same point are presented taking into account or ignoring shear deformation effect for various values of the loading factor μ as compared with those obtained from a FEM solution [MSC/NASTRAN for Windows] using 5600 solid elements (Fig.5c). From the obtained results, it is observed that the influence of the shear deformation effect is not so intense and could be ignored (as it was expected for a thin-walled beam), the significant influence of the nonlinear analysis effect is once more easily verified especially in the case of intense loading, while the convergence of the obtained results employing the proposed method, compared with those obtained from a 3-D FEM solution using solid elements is remarkable.

4.3 Example 3

To demonstrate the range of applications of the proposed method, a cantilever beam of length $l = 2.50m$ (Fig.6a), of the non-symmetric cross section ($A = 0.01186m^2$, $E = 2.1 \times 10^8 kN/m^2$, $\nu = 0.3$, $C_s = 5.232 \times 10^{-7} m^6$, $I_t = 1.910 \times 10^{-6} m^4$) of Fig. 6b has been studied. Since the proposed analysis has been developed with respect to the principal shear system of axes, the geometric, the inertia constants and the shear deformation coefficients of the examined cross section with respect to an original coordinate system $C\tilde{Y}\tilde{Z}$ (Fig. 6b) are presented in the first column of Table 8, while in the second column of this table the same constants are evaluated with respect to axes parallel to the principal shear coordinate system CYZ, resulting from the rotation of the $C\tilde{Y}\tilde{Z}$ system by the angle θ^S [Sapountzakis and Mokos (2005)]. In Tables 9, 10 the displacements w, v at the free end of the cantilever beam, subjected to a gradually increasing concentrated at the same point axial $\mu P_{\tilde{X}}$ ($P_{\tilde{X}} = -120kN$) and uniformly distributed axial $\mu p_{\tilde{X}}$ and transverse $\mu p_{\tilde{Z}}$ ($p_{\tilde{X}} = -20kN/m$, $p_{\tilde{Z}} = -40kN/m$) load are presented taking into account or ignoring shear deformation effect, for various values of the loading factor μ. The significant influence of the nonlinear analysis effect is once more verified especially in the case of intense loading, while the discrepancy of the results due to the influence of the shear deformation effect necessitates its inclusion in the analysis, especially for the calculation of the displacement w.

5 Concluding remarks

In this paper a boundary element method is developed for the nonlinear flexural – torsional analysis of Timoshenko beam-columns of arbitrary simply or multiply connected constant cross section, undergoing moderate large deflections under general boundary conditions. Seven boundary value problems are formulated with respect to the transverse displacements, to the axial displacement, to the angle of twist (which is assumed to be small), to the primary warping function and to two stress functions and solved using the Analog Equation Method, a BEM based method. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. The main conclusions that can be drawn from this investigation are

- The numerical technique presented in this investigation is well suited for computer aided analysis for beam-columns of arbitrary simply or multiply connected cross section.

- The convergence of the obtained results employing the proposed method, compared with those obtained from a 3-D FEM solution using solid elements is remarkable. Having in mind both the disadvantages of this solution (difficulties in support modelling, in discretizing a complex structure, in discretizing a structure including thin walled members (shear-locking, membrane-locking), in the increased number of degrees of freedom leading to severe or unrealistic computational time, in the reduced oversight of the 3-D FEM solution compared with that of the beam-like structures employing stress resultants) and the fact that the use of shell elements cannot give accurate results since the warping of the walls of a cross section cannot be taken into account (midline model), the importance of the proposed method becomes more evident.

- The significant influence of geometrical nonlinear analysis in beam elements subjected in intense transverse loading is verified.

- The discrepancy between the results of the linear and the nonlinear analysis demonstrates the significant influence of the axial loading.

- In some cases the remarkable increment of all the deflections due to the influence of the shear deformation effect demonstrates its significant influence in nonlinear analysis.

- The developed procedure retains the advantages of a BEM solution over a pure domain discretization method since it requires only boundary discretization.

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