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He's Homotopy Perturbation Method and Fractional Complex Transform for Analysis Time Fractional Fornberg-Whitham Equation

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ABSTRACT

In this article, time fractional Fornberg-Whitham equation of He's fractional derivative is studied. To transform the fractional model into its equivalent differential equation, the fractional complex transform is used and He's homotopy perturbation method is implemented to get the approximate analytical solutions of the fractional-order problems. The graphs are plotted to analysis the fractional-order mathematical modeling.

KEYWORDS

Time fractional Fornberg-Whitham equation; fractional complex transform; He's homotopy perturbation method

1 Introduction

Over the last few years, the study of the fractional calculus and applications in the area of life science, physics and the engineering has been paid a great attention. The fractional calculus are also used in many other fields, such as optics, solitary waves, control theory of dynamical systems, and so on, which can be derived by linear or nonlinear fractional order differential equations. Recently, the studies of nonlinear problems and their effects are of widely significance. Many analytical and approximation methods have been presented to solve nonlinear fractional differential equations such as [1–11]. The homotopy perturbation method (HPM) [12–14] is widely applied to various science and engineering problems. This method was first proposed by He [12]. Fractional complex transform was suggested also by He et al. [15–19], which converts the fractional differential equation into its equivalent differential equation, so that the HPM can be effectively used. Now it is considered as a powerful method to find the approximation solutions of nonlinear fractional order differential equations.

In this paper, we study the time fractional Fornberg-Whitham (FW) equation [20] as follows:

$$\frac{\partial^\beta u}{\partial t^\beta} - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial u}{\partial x} = u \frac{\partial^3 u}{\partial x^3} - u \frac{\partial u}{\partial x} + 3 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} \quad t > 0, \quad 0 < \beta \leq 1 \quad (1)$$



with the initial condition:

$$u_0(x, 0) = e^{\frac{x}{2}} \quad (2)$$

here β is the fractal dimensions of the fractal medium and $\frac{\partial^\beta}{\partial t^\beta}$ is He's fractional derivative defined [21–23]:

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{1}{\Gamma(n - \beta)} \frac{d^n}{dt^n} \int_{t_0}^t (s - t)^{n-\beta-1} [u_0(s) - u(s)] ds \quad (3)$$

where $u_0(x, t)$ is the solution of its starting point of the nonlinear fractional order model. When $\beta = 1$, Eq. (1) becomes to be the original Fornberg-Whitham (FW) model which is a significant mathematical equation in mathematical physics. This equation was presented by Parkes et al. [24–26] in 1978 which describes nonlinear water waves with peakon solutions. The peakon solution is a special solitary wave solution which is peaked in the limiting case. The FW equation has been found to require peakon results as a simulation for limiting wave heights as well as the frequency of wave breaks. Now many scholars have researched the FW model for the fractional-order derivative because of its fractional calculus applications.

2 HPM Procedure

To illustrate the basic ideas of this method [27–32], we consider the following nonlinear functional equation:

$$A(u) - f(r) = 0, \quad r \in \Omega \quad (4)$$

with the following boundary conditions:

$$B(u - \frac{\partial u}{\partial n}) = 0, \quad r \in \Lambda \quad (5)$$

where A is a general functional operator, B is a boundary operator, $f(r)$ is a known analytical function, and Λ is the boundary of the domain Ω . The operator A can be decomposed into two operators L and N , where L is linear, and N is nonlinear operator. Eq. (4) can be written as follows:

$$L(u) + N(u) - f(r) = 0 \quad (6)$$

Using the homotopy technique, we construct a following homotopy:

$$H(\xi, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \quad p \in [0, 1], \quad r \in \Omega \quad (7)$$

where the homotopy parameter p is considered as a small parameter.

Obviously, we can get:

$$\begin{aligned} H(\xi, 0) &= L(\xi) - L(\xi_0) \\ H(\xi, 1) &= A(\xi) - f(r) \end{aligned} \quad (8)$$

The HPM uses embedding parameter p as an expanding parameter, and basic assumption is that the solution of Eq. (6) can be specified as a power series in p :

$$\xi = \xi_0 + p\xi_1 + p^2\xi_2 + p^3\xi_3 + \dots \quad (9)$$

When $p \rightarrow 1$, the approximate solution of Eq. (1):

$$u = \lim_{p \rightarrow 1} \xi = \xi_0 + \xi_1 + \xi_2 + \xi_3 + \dots \quad (10)$$

3 HPM Implementation

According to the HPM, the first step is to transform the fractional model equation into its equivalent differential equation:

$$y = \frac{t^\beta}{\Gamma(1 + \beta)} \quad (11)$$

Eq. (1) can be written into its equivalent differential equation form:

$$\frac{\partial u}{\partial y} - \frac{\partial^3 u}{\partial x^2 \partial y} + \frac{\partial u}{\partial x} - u \frac{\partial^3 u}{\partial x^3} + u \frac{\partial u}{\partial x} - 3 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} = 0 \quad (12)$$

Applying the HPM process, substituting Eq. (11) into Eq. (7), we have a set of equations that is to be simultaneously solved:

$$\frac{\partial u_0}{\partial y} - e^{\frac{x}{2}} = 0, \quad u_0(x, 0) = e^{\frac{x}{2}} \quad (13)$$

$$\frac{\partial u_1}{\partial y} - \frac{\partial^3 u_0}{\partial x^2 \partial y} + \frac{\partial u_0}{\partial x} - u_0 \frac{\partial^3 u_0}{\partial x^3} + u_0 \frac{\partial u_0}{\partial x} - 3 \frac{\partial u_0}{\partial x} \frac{\partial^2 u_0}{\partial x^2} = 0, \quad u_1(x, 0) = 0 \quad (14)$$

$$\begin{aligned} \frac{\partial u_2}{\partial y} - \frac{\partial^3 u_1}{\partial x^2 \partial y} + \frac{\partial u_1}{\partial x} - u_0 \frac{\partial^3 u_1}{\partial x^3} - u_1 \frac{\partial^3 u_0}{\partial x^3} + u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} - 3 \frac{\partial u_0}{\partial x} \frac{\partial^2 u_1}{\partial x^2} - \\ - 3 \frac{\partial u_1}{\partial x} \frac{\partial^2 u_0}{\partial x^2} = 0, \quad u_2(x, 0) = 0 \end{aligned} \quad (15)$$

$$\begin{aligned} \frac{\partial u_3}{\partial y} - \frac{\partial^3 u_2}{\partial x^2 \partial y} + \frac{\partial u_2}{\partial x} - u_0 \frac{\partial^3 u_2}{\partial x^3} - u_1 \frac{\partial^3 u_1}{\partial x^3} - u_2 \frac{\partial^3 u_0}{\partial x^3} + u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_0}{\partial x} - 3 \frac{\partial u_0}{\partial x} \frac{\partial^2 u_2}{\partial x^2} - \\ - 3 \frac{\partial u_1}{\partial x} \frac{\partial^2 u_1}{\partial x^2} - 3 \frac{\partial u_2}{\partial x} \frac{\partial^2 u_0}{\partial x^2} = 0, \quad u_3(x, 0) = 0 \end{aligned} \quad (16)$$

We can get a solution for the Eqs. (13)–(16) in the form:

$$u_0(x, y) = e^{\frac{x}{2}}(y + 1) \quad (17)$$

$$u_1(x, y) = -e^{\frac{x}{2}}\left(\frac{1}{4}y + \frac{1}{4}y^2\right) \quad (18)$$

$$u_2(x, y) = e^{\frac{x}{2}}\left(\frac{1}{24}y^3 - \frac{1}{16}y\right) \quad (19)$$

$$u_3(x, y) = -e^{\frac{x}{2}}\left(\frac{1}{192}y^4 - \frac{1}{96}y^3 - \frac{1}{64}y^2 + \frac{1}{64}y\right) \quad (20)$$

Proceeding in the above manner, the rest of the components can be obtained. Thus we got the approximate solution which has the following form:

$$u_{approx}(x, y) = e^{\frac{x}{2}} \left(1 + \frac{43}{64}y - \frac{15}{64}y^2 + \frac{5}{96}y^3 - \frac{1}{192}y^4 + \dots \right) \quad (21)$$

Substituting Eq. (11) in Eq. (21), we get:

$$u_{approx}(x, t) = e^{\frac{x}{2}} \left(1 + \frac{43}{64} \left[\frac{t^\beta}{\Gamma(1+\beta)} \right] - \frac{15}{64} \left[\frac{t^\beta}{\Gamma(1+\beta)} \right]^2 + \frac{5}{96} \left[\frac{t^\beta}{\Gamma(1+\beta)} \right]^3 - \frac{1}{192} \left[\frac{t^\beta}{\Gamma(1+\beta)} \right]^4 + \dots \right) \quad (22)$$

4 Numerical Results and Discussion

The exact result [20] of Eq. (1):

$$u(x, t) = e^{\left(\frac{x}{2} - \frac{2}{3}t\right)} \quad (23)$$

In Figs. 1 and 2, the approximate solution obtained by fractional complex transform and HPM and actual solution of Eq. (1) are plotted. It is observed that HPM solutions are in closed contact with the exact solution. In Figs. 3 and 4, the solutions of Eq. (1) at various fractional-order of the derivatives are plotted which shows that the balancing phenomena between dispersion and nonlinearity is valid.

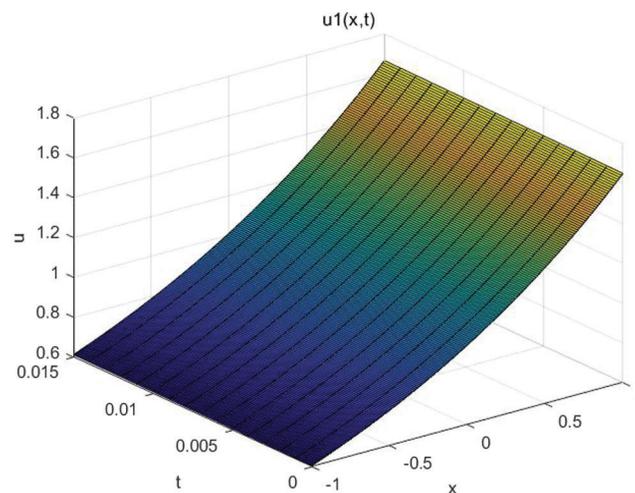


Figure 1: Approximate solution of Eq. (1) for $\beta = 1$

In Figs. 5 and 6, the the approximate solutions of Eq. (1) at different fractional order of the derivatives are plotted. This behavior illustrated that there is directly relationship between both the width and the height of the solitary wave and the value of β . We can change the shape of solitary wave with the help of adjusting the value of fractal order β are plotted. In Figs. 7 and 8, the numerical results illustrated which confirms the high accuracy of the method.

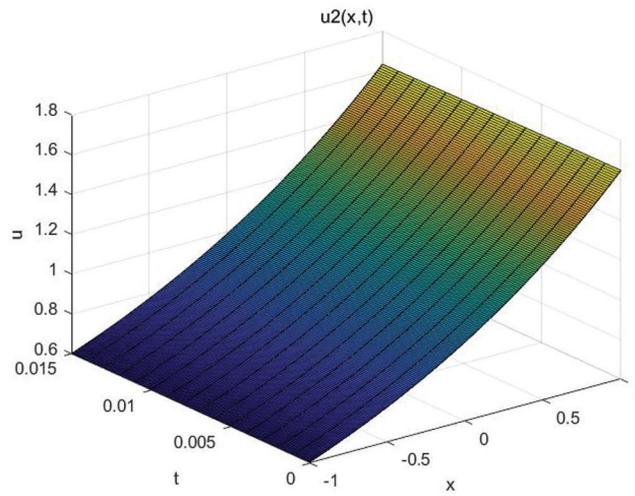


Figure 2: Exact solution of Eq. (1) for $\beta = 1$

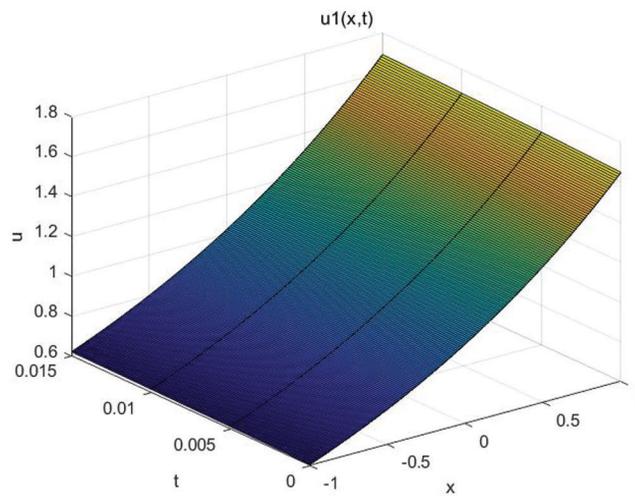


Figure 3: Approximate solution of Eq. (1) for $\beta = 0.8$

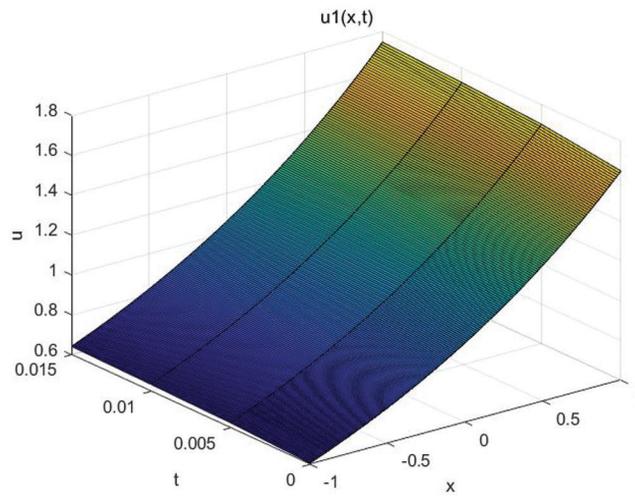


Figure 4: Approximate solution of Eq. (1) for $\beta = 0.6$

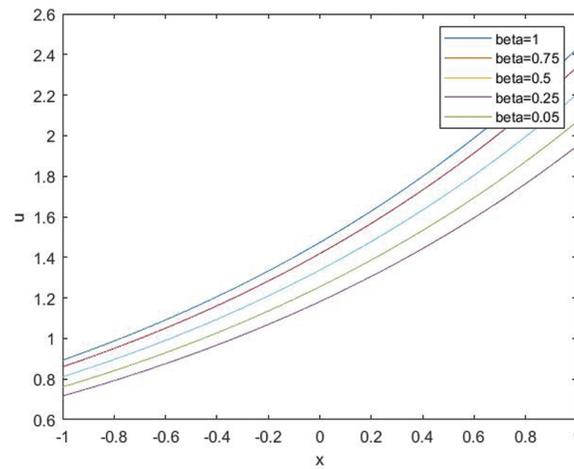


Figure 5: The relationship between β and amplitude of Eq. (1) for $t = 0.3$

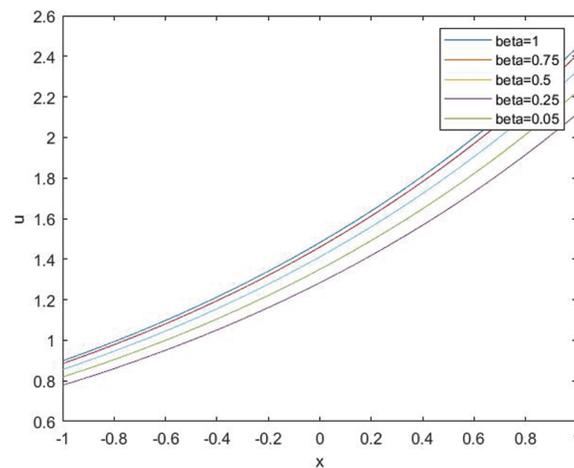


Figure 6: The relationship between β and amplitude of Eq. (1) for $t = 0.5$

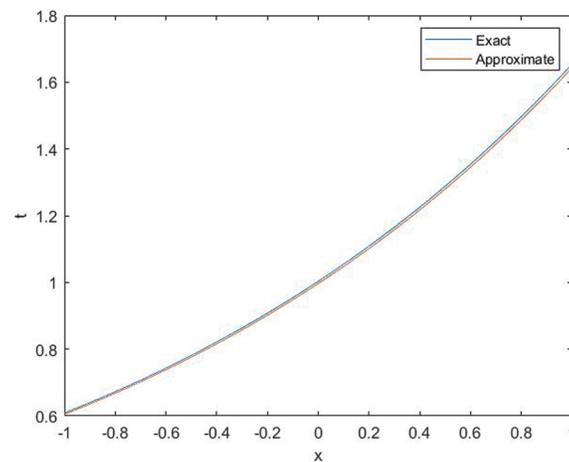


Figure 7: The relationship between exact solution (blue) and approximate solution (red) of Eq. (1) for $\beta = 1, t = 0.005$

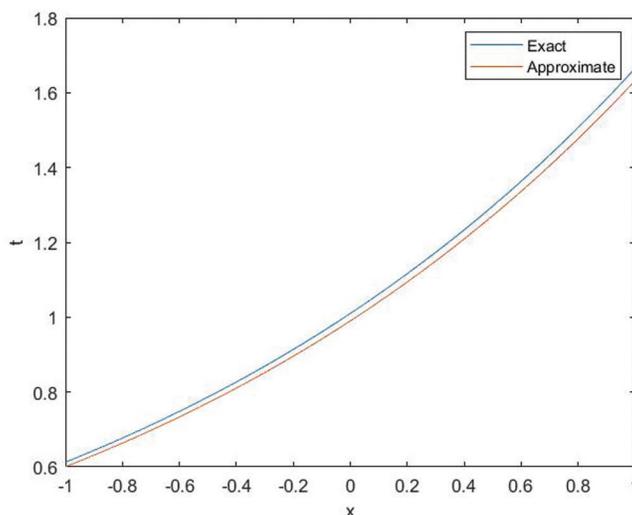


Figure 8: The relationship between exact solution and approximate solution (red) of Eq. (1) for $\beta = 1$, $t = 0.015$

5 Conclusion

In this manuscript, a mathematical technology is used to find the solution of time fractional Fornberg-Whitham equation. The fractional-derivatives are discussed within He's fractional derivative. The solutions are determined for fractional-order problems which shows the high accuracy and efficiency. This is easy and can be extended to other nonlinear differential equations with fractal derivatives in science and engineering.

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